MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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6.050J/2.110J **Information and Entropy** **Spring 2006**

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Problem Set 12 Solutions

Due: May 12, 2006

Solution to Problem 1: When energy becomes really expensive..

Solution to Problem 6, part c.

You must run it in the reverse direction.

Solution to Problem 1, part b.

The temperature of outside is $T_1 = 268.15$ degrees, and the room temperature is $T_2 = 294.15$ degrees Kelvin.

Solution to Problem 1, part c.

To find a relationship between T_1, T_2, H_c, and H_d we take the equation given in th eproblem statement

$$TdS = \left(\sum_i p_i (E_i(H) - E)^2 \right) \frac{1}{k_B T} \left(\frac{1}{T} dT - \frac{1}{H} dH \right) \tag{12–1}$$

since $dS = 0$ the equation reduces to

$$\frac{1}{T} dT = \frac{1}{H} dH$$

integrating from c to d we have

$$\begin{aligned} \int_{T_2}^{T_1} \frac{1}{T} dT &= \int_{H_c}^{H_d} \frac{1}{H} dH \\ \ln \left(\frac{T_1}{T_2} \right) &= \ln \left(\frac{H_d}{H_c} \right) \\ \frac{T_1}{T_2} &= \frac{H_d}{H_c} \end{aligned} \tag{12–2}$$

Solution to Problem 1, part d.

Thus finding H_d we have

$$\begin{aligned} H_d &= H_c \frac{T_1}{T_2} \\ &= 1000 \times \frac{268.15}{294.15} \\ &= 911.6 \text{ A/m} \end{aligned} \tag{12–3}$$

$$\tag{12–4}$$

Solution to Problem 1, part e.

The heat extracted from the cage is

$$Q = (S_2 - S_1)T_1 \tag{12–5}$$

The work done on the system is the heat pumped to the warm environment less the heat extracted from the cold environment

$$W = \frac{S_2 - S_1}{T_2 - T_1} \tag{12–6}$$

The coefficient of performance then is

$$\begin{aligned} \eta &= \frac{T_1}{T_2 - T_1} \\ &= \frac{268.15}{294.15 - 268.15} \\ &= 10.31 \end{aligned} \tag{12–7}$$

Solution to Problem 1, part f.

Again, to find a relationship between T_1, T_2, H_a, and H_b we take the equation given

$$TdS = \left(\sum_i p_i (E_i(H) - E)^2 \right) \frac{1}{k_B T} \left(\frac{1}{T} dT - \frac{1}{H} dH \right) \tag{12–8}$$

but once more, since $dS = 0$, the equation reduces to

$$\frac{1}{T} dT = \frac{1}{H} dH$$

integrating from a to b we have

$$\begin{aligned} \int_{T_1}^{T_2} \frac{1}{T} dT &= \int_{H_a}^{H_b} \frac{1}{H} dH \\ \ln\left(\frac{T_2}{T_1}\right) &= \ln\left(\frac{H_b}{H_a}\right) \\ \frac{T_2}{T_1} &= \frac{H_b}{H_a} \end{aligned} \tag{12–9}$$

Solution to Problem 1, part g.

The magnetic field H_a is

$$\begin{aligned} H_a &= H_b \frac{T_1}{T_2} \\ &= 2500 \times \frac{268.15}{294.15} \\ &= 2279.02 \text{ A/m} \end{aligned} \tag{12–10}$$

(12–11)

Solution to Problem 1, part h.

Since S is constant in this (adiabatic) leg, $dq = 0$.

To go further you have to calculate the probabilities, since you need them to find the energy E at each of the four corners. You already know the temperature and magnetic field at each corner, so it is straightforward to find α and then the probabilities using these equations from Chapter 12:

$$p_i = e^{-\alpha} e^{-E_i/k_B T} \tag{12–12}$$

$$\alpha = \ln\left(\sum_i e^{-E_i/k_B T}\right) \tag{12–13}$$

Solution to Problem 1, part i.

For corners a and b:

$$p_{up} = \frac{e^{m_{eff} H_a/k_B T_1}}{e^{m_{eff} H_a/k_B T_1} + e^{-m_{eff} H_a/k_B T_1}} \tag{12–14}$$

First calculate the exponential.

$$\begin{aligned} e^{m_{eff} H_a/k_B T_1} &= \exp\left(\frac{1.165 \times 10^{-22} \times 2279.02}{268.15 \times 1.38 \times 10^{-23}}\right) \\ &= 1.4455 \times 10^{31} \end{aligned} \tag{12–15}$$

Thus...

$$\begin{aligned} p_{up,a,b} &= \frac{1.4455 \times 10^{31}}{1.4455 \times 10^{31} + \frac{1}{1.4455 \times 10^{31}}} \\ &= 1 - 4.79 \times 10^{-63} \end{aligned} \tag{12–16}$$

$$\begin{aligned} p_{down,a,b} &= 1 - p_{up} \\ &= 4.79 \times 10^{-63} \end{aligned} \tag{12–17}$$

For corners c and d:

$$p_{up} = \frac{e^{m_{eff} H_d/k_B T_1}}{e^{m_{eff} H_d/k_B T_1} + e^{-m_{eff} H_d/k_B T_1}} \tag{12–18}$$

First calculate the exponential...

$$\begin{aligned} e^{m_{eff} H_d/k_B T_1} &= \exp\left(\frac{1.165 \times 10^{-22}}{\times} 911.6268.15 \times 1.38 \times 10^{-23}\right) \\ &= 2.9108 \times 10^{12} \end{aligned} \tag{12–19}$$

Thus...

$$\begin{aligned} p_{up,c,d} &= \frac{2.9108 \times 10^{12}}{2.9108 \times 10^{12} + \frac{1}{2.9108 \times 10^{12}}} \\ &= 1 - 1.18 \times 10^{-25} \end{aligned} \tag{12–20}$$

$$\begin{aligned} p_{down,c,d} &= 1 - p_{up} \\ &= 1.18 \times 10^{-25} \end{aligned} \tag{12–21}$$

Solution to Problem 1, part j.

$$\begin{aligned}
E_a &= \sum_i E_i p_i \\
&= -m_{eff} H_a p_{up,a} + m_{eff} H_a p_{down,a} \\
&= m_{eff} H_a (p_{down,a} - p_{up,a}) \\
&= -2.655 \times 10^{-19} Joules && (12\text{–}22) \\
E_b &= \sum_i E_i p_i \\
&= -m_{eff} H_b p_{up,b} + m_{eff} H_b p_{down,b} \\
&= m_{eff} H_b (p_{down,b} - p_{up,b}) \\
&= -2.9125 \times 10^{-19} Joules && (12\text{–}23) \\
E_c &= \sum_i E_i p_i \\
&= -m_{eff} H_c p_{up,c} + m_{eff} H_c p_{down,c} \\
&= m_{eff} H_c (p_{down,c} - p_{up,c}) \\
&= -1.165 \times 10^{-19} Joules && (12\text{–}24) \\
E_d &= \sum_i E_i p_i \\
&= -m_{eff} H_d p_{up,d} + m_{eff} H_d p_{down,d} \\
&= m_{eff} H_d (p_{down,d} - p_{up,d}) \\
&= -1.062 \times 10^{-19} Joules && (12\text{–}25) \\
& && (12\text{–}26)
\end{aligned}$$

Solution to Problem 1, part k.

$$\begin{aligned}
S_1 &= k_B \sum_i p_i \ln\left(\frac{1}{p_i}\right) \\
&= k_B \left(p_{up,a,b} \ln\left(\frac{1}{p_{up,a,b}}\right) + p_{down,a,b} \ln\left(\frac{1}{p_{down,a,b}}\right)\right) \\
&= 1.3673 \times 10^{-83} && (12\text{–}27) \\
S_2 &= k_B \sum_i p_i \ln\left(\frac{1}{p_i}\right) \\
&= k_B \left(p_{up,c,d} \ln\left(\frac{1}{p_{up,c,d}}\right) + p_{down,c,d} \ln\left(\frac{1}{p_{down,c,d}}\right)\right) \\
&= 1.3488 \times 10^{-46} && (12\text{–}28)
\end{aligned}$$

therefore

$$S_2 - S_1 = 1.3488 \times 10^{-46} \text{ Joules/Kelvin} \quad (12\text{–}29)$$

Solution to Problem 1, part l.

$$\begin{aligned}
dq_{ba} &= TdS \\
&= 0 \text{ Joules} && (12\text{–}30) \\
dq_{ad} &= TdS \\
&= T_1(S_2 - S_1) \\
&= 3.6168 \times 10^{-44} \text{ Joules} && (12\text{–}31) \\
dq_{dc} &= TdS \\
&= 0 \text{ Joules} && (12\text{–}32) \\
dq_{cb} &= TdS \\
&= T_2(S_1 - S_2) \\
&= -3.9675 \times 10^{-44} \text{ Joules} && (12\text{–}33) \\
& && (12\text{–}34)
\end{aligned}$$

Solution to Problem 1, part m.

$$\begin{aligned}
dw_{ba} &= dE_{ba} - dq_{ba} \\
&= E_a - E_b - 0 \\
&= 2.5747 \times 10^{-20} \text{ Joules} \\
dw_{ad} &= dE_{ad} - dq_{ad} \\
&= E_d - E_a - dq_{ad} \\
&= 1.593 \times 10^{-19} \text{ Joules} \\
dw_{dc} &= dE_{dc} - dq_{dc} \\
&= E_c - E_d - 0 \\
&= -1.0299 \times 10^{-20} \text{ Joules} \\
dw_{cb} &= dE_{cb} - dq_{cb} \\
&= E_b - E_c - dq_{cb} \\
&= -1.7475 \times 10^{-19} \text{ Joules}
\end{aligned} \tag{12–35}$$

Solution to Problem 1, part n.

The work is

$$w = -(dq_{ba} + dq_{cb}) = 3.5069 \times 10^{-45}\text{Joules} \tag{12–36}$$

Solution to Problem 1, part o.

$$10.31 \tag{12–37}$$

This is exactly the same as the coefficient.

Solution to Problem 1, part p.

The number of Joules required to heat one gram of air one degree is

$$\frac{0.715}{3.9675 \times 10^{-44}} = 1.8022 \times 10^{43} \text{ cycles} \tag{12–38}$$

Solution to Problem 2: Information is Cool

Solution to Problem 2, part a.

$$\frac{75 \text{ Calories/hour} \times 4.1868 \times 10^3 \text{ Joules/Calorie}}{3600 \text{ sec/hour}} = 87.225 \text{ Joules/sec} \tag{12–39}$$

People don't light up like lightbulbs because the energy they expend is distributed about the whole body, not concentrated on a microscopic filament.

Under these assumptions, a glowworm would dissipate 30Watts=30J/s, hence 108,000J/h. Assuming the glowworm kept all the heat within itself and that at the beginning it was at the temperature of its surroundings,to compute the gain in temperature we use

$$Q = mSp\Delta T \rightarrow \Delta T = \frac{Q}{mSp} \tag{12–40}$$

mis 100g, Q is 108,000J=25795 Cal, and Sp is 1 Cal/g/K. So $\Delta T = 257K$. (This means that assuming the glowworm was in origin at a temperature of $300K = 27°C = 80.33°F$ after one hour it ended up being at a temperature of $554K = 283°C = 542°F$, quite a warm up for just one hour. We can safely discard the idea that glowworms are lit by lightbulbs.

Solution to Problem 2, part b.

A person needs to consume 75 Calories times 24 hours, or

$$75 \text{ Calories/hour} \times 24 \text{ hours/day} = 1800 \text{ Calories/day} \tag{12–41}$$

If the brain requires 20% of the energy consumed by the body, then 360 Calories/day are devoted to "thinking".

Solution to Problem 2, part c.

Seth burns an extra 931 Calories per week than a person who does no exercise. Paul burns an extra 1132 Calories per week than a person whoe does no exercise, however since he replenishes 3 times a week with an extra 260 Calories every time he jogs, he ends up only burning an excess of 352.5 Calores with respect to a person who does no exercise. To remain at a constant weight, Paul should burn an extra 578.5 Calories per week or eat only 3/4 of a serving of the ice cream per week. Paul thus gains, in a year:

$$578.5 \text{ Calories/week} \times 52 \text{ weeks/year} = 30082 \text{ Calories/year} \tag{12–42}$$

Danial gains:

$$931 \text{ Calories/week} \times 52 \text{ weeks/year} = 48412 \text{ Calories/year} \tag{12–43}$$

Thus Paul gains:

$$\frac{30082 Calories \times 4.1868 \times 10^3 \text{ Joules/Calorie}}{33.1 \times 10^6 \text{ Joules/kg fat}} = 3.8 \text{ kg fat} \quad (12\text{–}44)$$

And John gains:

$$\frac{48412 Calories \times 4.1868 \times 10^3 \text{ Joules/Calorie}}{33.1 \times 10^6 \text{ Joules/kg fat}} = 6.12 \text{ kg fat} \quad (12\text{–}45)$$

Solution to Problem 2, part d.

The amount of heat the dorm is losing, in Watts, is:

$$\frac{2.052 \times 10^8 \text{Joules/hour}}{3600 \text{ sec/hour}} = 57000 \text{ Watts} \quad (12\text{–}46)$$

If the temperature is to remain constant, the students must produce the same amount of energy. Since we have no information about the proportion of students awake and asleep, we assume all undergraduates are either solving problem sets or awake, since it seems that undergraduates at MIT prefer to sleep in class

$$250P + 100A = 57000 \text{ Watts} \quad (12\text{–}47)$$

where A = Awake and P = doing problem sets, If the dorm has 300 students on average then the sum of A and P equals 300 so

$$\begin{aligned} 250P + 100(300 - P) &= 57000 \\ 30000 + 150P &= 57000 \\ P &= 180 \text{ students} \end{aligned} \quad (12\text{–}48)$$

We conclude that MIT should have around 180 undergraduate students out of 300 solving problem sets at any time of the day to compensate for heat lost to the outdoors.

We can use the problem of maximum entropy to infer the proportion of students awake, asleep and solving problem sets without having to assume anything about th ehabits of sleep of MIT undergraduates. We have two constraints: (1) The heat each student must dissipate on average to compensate for the loss to the outdoors, and (2) the probability constraint. And we have three variables, so the problem can be solved using the simple approach replacing variables and finding then th emaximum. If you do it, you will obtain the following distribution

P -¿ 0.632606, S -¿ 0.204366, A -¿ 0.163028

Solution to Problem 2, part e.

The number of Calories consumed in raising 330 ml of water to body temperature (37 degrees Celsius) is

$$0.330 \text{ Liter} \times 1 \text{ Calories/Liter/degree C} \times 37 \text{degrees C} = 12.2 \text{ Calories} \quad (12\text{–}49)$$

Only 7% of the Calories are consumed raising the rootbeer to body temperature. So Paul's argument is not correct.