Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
Department of Mechanical Engineering

6.050J/2.110J **Information and Entropy** **Spring 2006**

Issued:

Problem Set 11 Solutions

Due:

Solution to Problem 1: Entropy Goes Up

Solution to Problem 1, part a.

The expectation of system energy E_s (the expected value of the energy) is calculated by the following formula.

$$\begin{aligned} E_s &= \sum_i p_{s,i} E_{s,i}(H) \\ &= 0.75\, m_d H - .25\, m_d H \\ &= 0.5\, m_d H \end{aligned} \tag{11–1}$$

Solution to Problem 1, part b.

The expectation of the environment energy E_e is found in a similar manner.

$$\begin{aligned} E_e &= \sum_j p_{e,j} E_{e,j}(H) \\ &= 0\, m_d H - 1\, m_d H \\ &= -m_d H \end{aligned} \tag{11–2}$$

Solution to Problem 1, part c.

The environment entropy is calculated via the following formula.

$$\begin{aligned} S_e &= k_B \sum_j p_{e,j} \ln\left(\frac{1}{p_{e,j}}\right) \\ &= 0. \end{aligned} \tag{11–3}$$

Solution to Problem 1, part d.

The system entropy S_s is calculated in a similar fashion.

$$\begin{aligned} S_s &= k_B \sum_i p_{s,i} \ln\left(\frac{1}{p_{s,i}}\right) \\ &= k_B \left(0.75 \ln\left(\frac{1}{0.75}\right) + 0.25 \ln\left(\frac{1}{0.25}\right)\right) \\ &= 0.56 k_B. \end{aligned} \tag{11–4}$$

Solution to Problem 1, part e.

The dimension of Boltzmann's Constant (k_B) is [Energy]/[Temperature], which in the International System of units corresponds to Joules per Kelvin. To determine the units of $\alpha$and β we look at the formula to determine probabilities by maximum entropy:

$$p_i = e^{\alpha} e^{\beta E_i} \tag{11–5}$$

α must be adimensional since it is in the exponent of an exponential function. There are several ways to figure out the units of β. The most inmediate one is to use a similar argument, and claim that if the exponent βE_i must be adimensional then β must have units of inverse energy. This is consistent with the definition of $\beta = 1/k_B T$, by dimensional analysis:

$$\beta = \frac{1}{k_B T} \longrightarrow \frac{1}{[\text{Energy}]} = \frac{1}{\frac{[\text{Energy}]}{[\text{Temperature}]}[\text{Temperature}]} \tag{11–6}$$

Solution to Problem 1, part f.

No energy leaves the system and environment combined (by definition) so the expectation of the total energy is just the sum of the expectations of the energy of the system and environment.

$$\begin{aligned} E_t &= E_s + E_e \\ &= 0.5\, m_d H - m_d H && (11\text{–}7) \\ &= -0.5\, m_d H && (11\text{–}8) \end{aligned}$$

Solution to Problem 1, part g.

To find β_t we can combine equations 11.13 and 11.17 from the notes

$$\textstyle\sum_i dp_i = 0 \quad dp_i = -p_i(E_i - E)d\beta \tag{11–9}$$

to deduce that

$$0 = \sum_{i,j} dp_{i,j} = -\sum_{i,j} p_{i,j}(E_{i,j} - E_t)d\beta \tag{11–10}$$

hence

$$\sum_{i,j} p_{i,j}(E_{i,j} - E_t) = 0 \tag{11–11}$$

Note that $E_{i,j} \in \{-2, 0, 0, or 2\} \times m_d H$.

$$\begin{aligned} 0 &= \sum_{i,j} (E_{i,j} - E_t)\, e^{-\beta_t E_{i,j}} \\ &= \sum_{i,j} E_{i,j} e^{-\beta_t E_{i,j}} - E_t \sum_{i,j} e^{-\beta_t E_{i,j}} \\ &= m_d H \left(-2e^{2m_d H\beta_t} + 2e^{-2m_d H\beta_t} + 0.5e^{2m_d H\beta_t} + 0.5e^{-2m_d H\beta_t} + 1\right) && (11\text{–}12) \\ &= 1 - \frac{3}{2}e^{2m_d H\beta_t} + \frac{5}{2}e^{-2m_d H\beta_t} && (11\text{–}13) \\ & && (11\text{–}14) \end{aligned}$$

the last equation can be solved as a quadratic equation with the replacement $t = e^{2m_d H\beta_t}$. You will obtain two values of t but only one (the positive one) makes sense. The final result is

$$\beta_t = \frac{\ln(5/3)}{2m_d H} \tag{11–15}$$

Solution to Problem 1, part h.

The probabilities are defined as

$$p_{i,j} = \frac{e^{-\beta_t E_{i,j}}}{\sum_{i,j} e^{-\beta_t E_{i,j}}} \tag{11–16}$$

Thus

So

$$p_{0,0} = \frac{25}{64} \tag{11–17}$$
$$p_{0,1} = \frac{15}{64} \tag{11–18}$$
$$p_{1,0} = \frac{15}{64} \tag{11–19}$$
$$p_{1,1} = \frac{9}{64} \tag{11–20}$$
$$\tag{11–21}$$

Solution to Problem 1, part i.

The total entropy is

$$\begin{aligned} S_t &= k_B \sum_i p_i \ln\left(\frac{1}{p_i}\right) \\ &= 1.32 k_B \end{aligned} \tag{11–22}$$

which is higher than the original entropy, of the system $0.56k_B$

Solution to Problem 1, part j.

First let us infer from the four probabilities for the total configuration $p_{t,i,j}$ the probabilities for the two system states $p_{s,i}$.

The energy is

$$\begin{aligned} E_s &= \sum_j [\sum_i p_{s=j,i}] E_{s=j}(H) \\ &= [p_{0,0} + p_{0,1}]\, E_{s=0} + [p_{1,0} + p_{1,1}]\, E_{s=1} \qquad = -0.25\, m_d H \end{aligned} \tag{11–23}$$

Thus we see that exactly half the total energy is in the system.

Solution to Problem 1, part k.

The system started out with $0.5m_dH$ Joules in it, and ended up with $-0.25m_dH$ Joules in it. Thus $0.75m_dH$ Joules flowed, but because of the sign, they did not flow from environment to system but in the opposite direction.

Solution to Problem 1, part l.

Much like we computed the energy in the system after mixing, we can compute the entropy:

$$S_s = k_B \sum_j [\sum_i p_{s=j,i}] \ln \frac{1}{[\sum_i p_{s=j,i}]} \tag{11–24}$$

$$= k_B \left([p_{0,0} + p_{0,1}] \frac{1}{[p_{0,0} + p_{0,1}]} + [p_{1,0} + p_{1,1}] \frac{1}{[p_{1,0} + p_{1,1}]} \right) \tag{11–25}$$

$$= 0.66 k_B \tag{11–26}$$

And so, the entropy in the system has increased by $\Delta S = k_B(0.66 - 0.56) = 0.1k_B$.

Solution to Problem 2: Wind Power

Solution to Problem 2, part a.

See Figure 11–2.

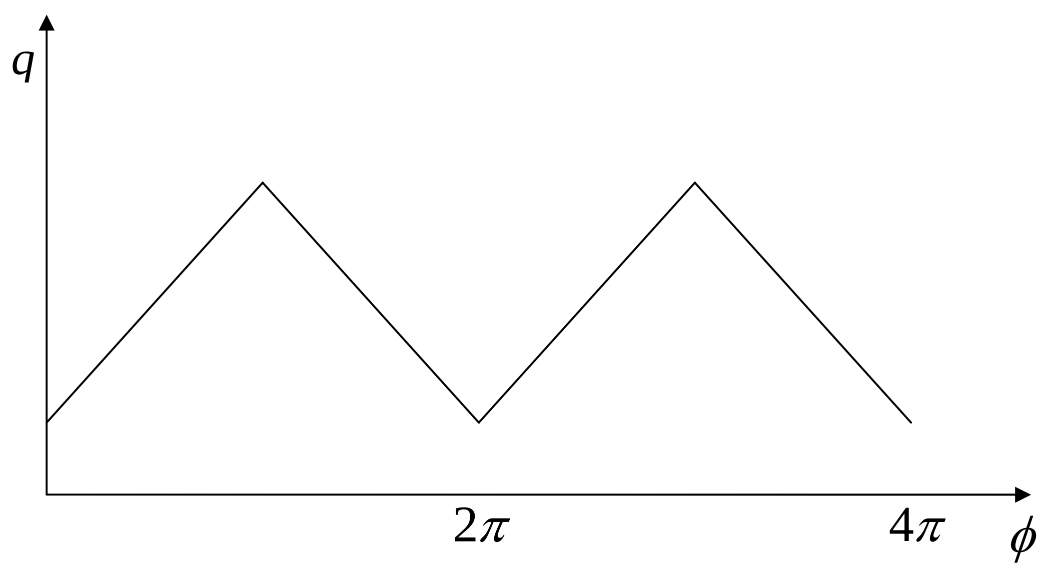

Figure 11–2: Diagram of charge as a function of ϕ.

Solution to Problem 2, part b.

See Figure 11–3.

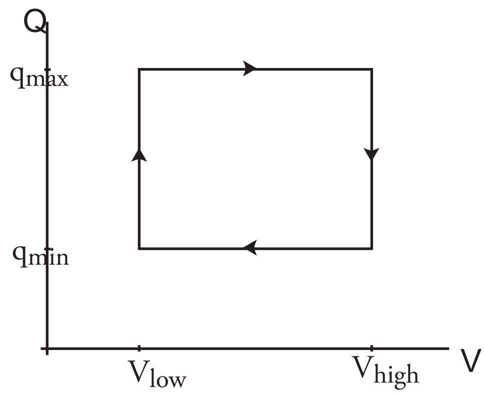

Figure 11–3: Charging cycle in the Charge-Voltage plane

Solution to Problem 2, part c.

See Figure 11–4.

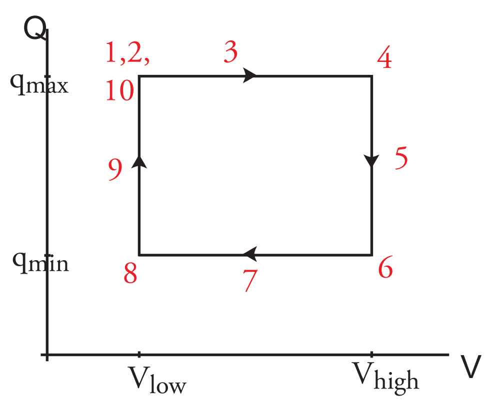

Figure 11–4: Anotated Charging cycle in the Charge-Voltage plane, as it would appear in the user's guide

Solution to Problem 2, part d.

See Figure 11–5.

Solution to Problem 2, part e.

The formula for the charge on a capacitor was given in the problem statement and is

$$q = \frac{\epsilon_0 RL\phi v}{d} \tag{11–27}$$

The initial position is $\phi = \pi$ and the plate is charged by

$$q_{max} = \frac{\epsilon_0 RL\pi v_{low}}{d} \tag{11–28}$$

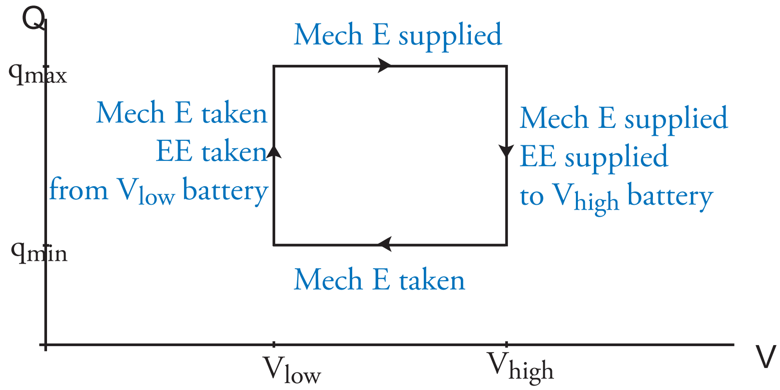

Figure 11–5: Anotated Charging cycle in the Charge-Voltage plane, energy flow detail

The minimum charge is the capacitor is

$$q_{min} = \epsilon_0 L v_{high} \tag{11–29}$$

The charge delivered is then the difference between the two

$$q_{delivered} = q_{max} - q_{min} = \frac{\epsilon_0 R L \pi v_{low}}{d} - \epsilon_0 L v_{high} \tag{11–30}$$

Solution to Problem 2, part f.

Using the result we just obtained for the charge supplied:

$$E_{high} = v_{high} \left(\frac{\epsilon_0 R L \pi v_{low}}{d} - \epsilon_0 L v_{high} \right) \tag{11–31}$$

Solution to Problem 2, part g.

Similarly, the low voltage battery delivers

$$E_{low} = v_{low} \left(\frac{\epsilon_0 R L \pi v_{low}}{d} - \epsilon_0 L v_{high} \right) \tag{11–32}$$

Solution to Problem 2, part h.

The mechanical energy supplied by the device is simply the difference between the last two computed energies

$$E_{mech} = E_{high} - E_{low} = (v_{high} - v_{low}) \left(\frac{\epsilon_0 R L \pi v_{low}}{d} - \epsilon_0 L v_{high} \right) \tag{11–33}$$

Solution to Problem 2, part i.

The charge delivered is

$$q_{delivered} = 1.15 \times 10^{-10}\, C \tag{11–34}$$

To charge by 10^{-6} coulombs, we require

$$\frac{10^{-6}}{1.15 \times 10^{-10}} = 8730 cycles \tag{11–35}$$

The conveyor belt runs at $10\, rev/s$, so we need

$$\frac{8730 cycles}{10 cycles/second} = 873\, seconds. \tag{11–36}$$