Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
Department of Mechanical Engineering

6.050J/2.110J **Information and Entropy** **Spring 2006**

Issued: May 2, 2006 **Due: May 5, 2006**

Problem Set 11

Problem 1: Entropy Goes Up

This problem is based on the magnetic dipole model in Chapter 11 of the notes. The system is assumed to have one dipole, as pictured, and for the purposes of this problem there is only one environment and it also contains exactly one dipole (this is to keep the calculations simple). For this problem you may assume that the environment and system have the same applied magnetic field.

The configuration is set up with the system having a high energy and the environment a low energy, and then the two are allowed to interact with the result that some energy in the form of heat may flow from the environment to the system. You will calculate the amount of heat and the entropy before and after this operation.

The configuration is shown in Figure 11–1.

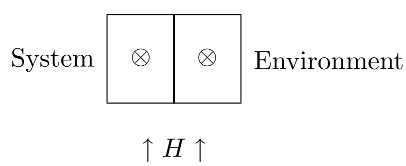

Figure 11–1: Dipole moment example.

Each of the dipoles shown can be either aligned with the field, in which case it contributes an energy $-m_d H$ or in the opposite direction, in which case it contributes $m_d H$. Thus the system has two states, one with energy $m_d H$, and one with energy $-m_d H$. The environment also has two states, with the same energies. As the problem starts, the two (system and environment) are isolated and each has a probability distribution as shown in these tables.

System				Environment			
State	Dipole	Energy $E_{s,i}(H)$	Probability $p_{s,i}$	State	Dipole	Energy $E_{e,j}(H)$	Probability $p_{e,j}$
$i = 0$	up	$-m_d H$	0.25	$j = 0$	up	$-m_d H$	1
$i = 1$	down	$m_d H$	0.75	$j = 1$	down	$m_d H$	0

Table 11–1: System and Environment Parameters

a. Find the system energy E_s (the expected value of the energy). You may leave your answer (and the other energies asked for) as a multiple of $m_d H$.

b. Find the environment energy E_e.

c. Find the environment entropy S_e. You may leave your answer (and the other entropies asked for) in terms of k_B.

d. Find the system entropy S_s.

e. Recall from the notes the units of k_B and entropy. What are the units of β and α?

Now consider what happens when the barrier between the system and the environment is removed, so they can interact. After some time has passed, your knowledge of the initial probabilities and separate energies is no longer relevant, and you can only treat the system as a whole. The Principle of Maximum Entropy can be used to estimate the probabilities of the four possible states (up-up, up-down, down-up, and down-down) without your assuming any information you do not have.

f. What is the total energy E_t?

g. Find β_t.

h. Find the four probabilities $p_{t,i,j}$.

i. Calculate the total entropy S_t and compare it to the sum of the system and environment entropies before the interaction.

j. Calculate the expected value of energy E_s in the system. (*Hint: You may find useful to look at this problem as you did for inference. After mixing, the information about the distribution of states in the system was lost. So it is difficult to compute the expected value of energy in the system, yet you have the information about the distribution of states in the total system, and using inference,... .*)

k. How much energy came into the system from the environment in the form of heat during the interaction?

l. What is the change in entropy of the system after mixing with the environment?.

Problem 2: Wind Power

Remote Possibility, Inc. makes equipment for pollution monitoring. Their products are deployed in remote locations without electric power. The energy to run the sensors and radio transmitters is obtained from the environment. To supplement solar cells, which do not work at night, they have developed a wind-driven battery charger. The CEO of the company has a personal dislike of magnetic fields (once he dropped a refrigerator magnet onto a floppy disk and lost all his data). He insisted that the charger use an electric field rather than a magnetic field. You are asked to analyze the design and determine if it will work, and if so how much energy can be converted from mechanical to electrical form. (Although this problem does not involve either information or entropy, it has the same approach to energy conversion that will be used in describing heat engines and the second law of thermodynamics.)

The device uses a windmill whose shaft, mounted on frictionless bearings, turns in the slightest breeze. The shaft and its housing have the patented design shown in cross section below. The housing is a plastic and steel cylindrical tube with the metal electrode on the top half. The shaft is also half plastic and half steel. The housing electrode has an electric charge q coulombs, and there is an equal and opposite charge $-q$ on the shaft electrode. The shaft turns to bring the two electrodes in and out of alignment. Let the shaft angle be denoted ϕ, as shown in Figure 11–1.

The shaft electrode is wired to a three-way switch (see the circuit below) that can connect it to a low-voltage battery with voltage V_{low} volts, a high-voltage battery with voltage V_{high} volts, or neither.

Your knowledge of electrostatics enables you to deduce the following.

- When the shaft is in partial alignment so there is an overlap of electrodes of ϕ radians, the charge q is approximately related to voltage v between the electrodes by the "parallel-plate capacitance" formula $q = \epsilon_0 LRv\phi/d$ where ϵ_0 is the permittivity of free space, 8.854×10^{-12} farads per meter, R is the radius of the shaft, L is the length of the shaft, and d is the gap between the electrodes. This formula is valid for $0 < \phi < \pi$.

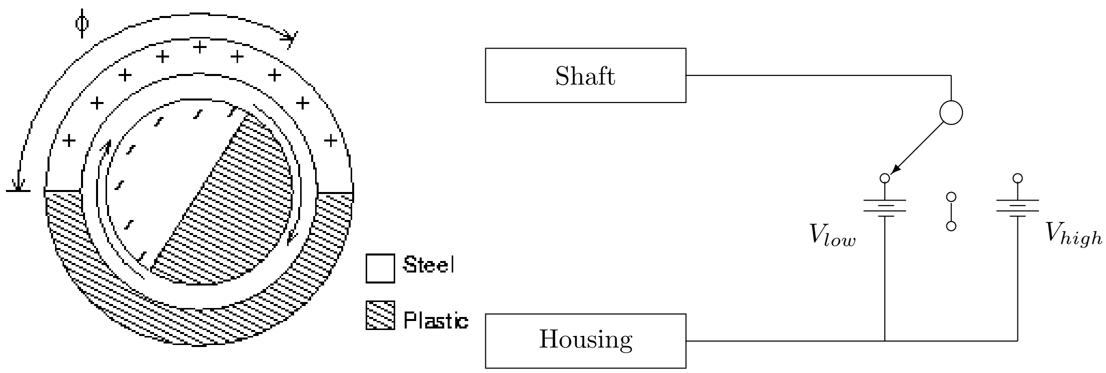

Figure 11–1: Charge pump

- For $\phi > \pi$ the charge comes back down toward 0. The charge is then a periodic function of ϕ with period 2π.
- There is an additional small charge on the electrodes even when they are away from each other, (due to stray capacitance) which you estimate to be $q = \epsilon_0 L v$. This charge is only significant when ϕ is near 0 or an integer times 2π.
- The charges on the electrodes exert a torque on the shaft that tends to bring the electrodes together.
- The charge q can only change if the switch connects the top plate with one of the batteries, in which case the energy supplied by the battery to change the charge from q_1 to q_2 is $v(q_2 - q_1)$.

The designer of the device describes the operation of charging the high-voltage battery as follows.

1. Start with the low-voltage battery connected and the electrodes in alignment, i.e., $\phi = \pi$.
2. Throw the switch to the middle position so the charge cannot change (the engineer knows how to design a circuit to throw the switch automatically in the final product).
3. As the shaft rotates (in either direction) the electrodes pull apart. Since the charge cannot change, the voltage rises. Continue until the voltage is equal to v_{high}.
4. Throw the switch so that the high-voltage battery is connected. Since the voltage is equal to the voltage of that battery this operation does not by itself affect q.
5. Let the shaft continue to rotate until the electrodes are opposite ($\phi = 0$). This causes the charge q to decrease, meaning charge is being delivered to the high-voltage battery.
6. Throw the switch to the middle position.
7. Let the shaft rotate so that the overlap of the electrodes increases. Since the charge cannot change, the voltage must go down. Continue until the voltage reaches v_{low}.
8. Throw the switch to connect to the low-voltage battery.
9. Let the shaft rotate until the electrodes are in alignment.

10. The shaft position and the charge are what they were at the beginning of these instructions, and some charge has been placed on the high-voltage battery thereby charging it up, and the same amount of charge has been taken from the low-voltage battery, thereby discharging it.

11. Repeat steps 2 - 9 as required to obtain the desired battery charge.

Naturally you want to know how much energy has been added to the high-voltage battery and how much was lost from the low-voltage battery, and therefore how much was supplied by the wind.

a. To start your analysis, plot the charge for a given voltage as a function of shaft position, between $\phi = 0$ and $\phi = 4\pi$ (two revolutions). Assume that d is less than R.

b. Draw the charging cycle as a rectangle in the charge-voltage plane (charge on the vertical axis, voltage on the horizontal).

c. Mark the parts of this diagram that correspond to the numbers in the description above.

d. Indicate for each of the four legs whether mechanical energy is being supplied to the device or taken from the device, and whether electrical energy is being supplied to or taken from either of the batteries.

e. Find a formula for the charge q_0 delivered to the high-voltage battery in one shaft rotation as a function of L, R, d, v_{low}, v_{high}, and ϵ_0.

f. Find the energy supplied to the high-voltage battery per cycle (this is the product of its voltage times the charge supplied).

g. Find the energy delivered by the low-voltage battery per cycle.

h. Find the energy supplied by the mechanical source per cycle.

i. By taking the device apart you discover that the shaft radius R is 3 cm, the length L is 10 cm, and the gap d is 1 mm. You guess that a moderate breeze could turn the shaft at a rate of 10 revolutions per second. You want to charge a 12-volt storage battery using a 1.5-volt battery for the low voltage. How long would it take to charge it with 10^{-6} coulombs (enough to run a 1 microwatt load for 12 seconds)?

Turning in Your Solutions

You may turn in this problem set by e-mailing your written solutions, M-files, and diary to 6.050-submit@mit.edu. Do this either by attaching them to the e-mail as text files, or by pasting their content directly into the body of the e-mail (if you do the latter, please indicate clearly where each file begins and ends). If you have figures or diagrams you may include them as graphics files (GIF, JPG or PDF preferred) attached to your email. Alternatively, you may turn in your solutions on paper in room 38-344. The deadline for submission is the same no matter which option you choose.

Your solutions are due 5:00 PM on Friday, May 5, 2006. Later that day, solutions will be posted on the course website.