Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
Department of Mechanical Engineering

6.050J/2.110J **Information and Entropy** **Spring 2006**

Issued:

Problem Set 10 Solutions

Due:

Solution to Problem 1: Well, Well, Well

Solution to Problem 1, part a.

Inside the well V(x) = 0 and therefore

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} \tag{10–2}$$

Solution to Problem 1, part b.

If E has a nonzero imaginary part E_{imag}, then the magnitude of $f(t)$ is a function of time, in particular

$$\mid f(t) \mid = exp(E_{imag} t/\hbar) \tag{10–3}$$

If $E_{imag} > 0$ then $\mid f(t) \mid$ gets large for large values of t (i.e., it blows up at infinity). If $E_{imag} < 0$ then $\mid f(t) \mid$ gets large for large values of $-t$ (i.e., it blows up at negative infinity). In either case it is impossible to normalize $\psi(x)$.

Solution to Problem 1, part c.

$$E\phi(x) = -\frac{\hbar^2}{2m}\frac{\partial^2 \phi(x,t)}{\partial x^2} \tag{10–4}$$

Solution to Problem 1, part d.

Since

$$\begin{aligned}
\phi(x) &= a\sin(kx) + b\cos(kx) && (10–5)\\
\frac{d\phi(x)}{dx} &= ak\cos(kx) - bk\sin(kx) && (10–6)\\
\frac{d^2\phi(x)}{dx^2} &= -ak^2\sin(kx) - bk^2\cos(kx)\\
&= -k^2\phi(x) && (10–7)
\end{aligned}$$

Therefore

$$E\phi(x) = \left(\frac{\hbar^2 k^2}{2m}\right)\phi(x) \tag{10–8}$$

so

$$E = \frac{\hbar^2 k^2}{2m} \tag{10–9}$$

Solution to Problem 1, part e.

One of the boundary conditions is $\phi(-L/2) = 0$, so

$$\begin{aligned} 0 &= \phi(L/2) \\ &= asin(kL/2) + bcos(kL/2) \end{aligned}$$

The other boundary condition is $\phi(L) = 0$, so

$$\begin{aligned} 0 &= \phi(-L/2) \\ &= a\sin(-kL/2) + b\cos(-kL/2) \\ &= -a\sin(kL/2) + b\cos(kL/2) \end{aligned}$$

Solution to Problem 1, part f.

Adding and subtracting the two equations, we find that both $a\sin(kL/2)$ and $b\cos(kL/2)$ should be zero; hence, for nonzero a and b:

$$\begin{aligned} \sin(kL/2) &= 0 \Longrightarrow \quad k_j = (\pi/L)j \qquad j = 2n \\ \cos(kL/2) &= 0 \Longrightarrow \quad k_j = (\pi/L)j \qquad j = 2n+1 \end{aligned}$$

that clearly cannot be satisfied simultaneously. For odd j, the cosine term is zero, so a must be zero. For even j, b must be zero. The values of a and b sorresponding to each k_j can then be determined normalizing $\phi(x)$.

Solution to Problem 1, part g.

$$e_j = \frac{\hbar^2 \pi^2 j^2}{2mL^2} \tag{10–10}$$

Solution to Problem 1, part h.

$$\phi_j(x) = \begin{cases} b\cos\left(\frac{j\pi x}{L}\right) & j \text{ odd} \\ a\sin\left(\frac{j\pi x}{L}\right) & j \text{ even} \end{cases} \tag{10–11}$$

Solution to Problem 1, part i.

$$e_1 = \frac{\hbar^2 \pi^2}{2mL^2} \tag{10–12}$$

Solution to Problem 1, part j.

$$e_2 = 2 \times \frac{\hbar^2 \pi^2}{mL^2} \tag{10–13}$$

Solution to Problem 1, part k.

$$\begin{aligned} e_1 &= \frac{\hbar^2 \pi^2}{2mL^2} && (10\text{–}14) \\ &= \frac{(1.054 \times 10^{-34}\text{Joule-seconds})^2 \times (3.1416)^2}{2 \times (9.109 \times 10^{-31}\text{kilograms}) \times (1.5 \times 10^{-9}\text{meters})^2} && (10\text{–}15) \\ &= 2.71 \times 10^{-22}\text{Joules} && (10\text{–}16) \\ & && (10\text{–}17) \end{aligned}$$

Solution to Problem 1, part l.

Express this ground-state energy in electron-volts (1 eV= 1.602×10^{-19} Joules).

$$\begin{aligned} e_1 &= 2.71 \times 10^{-22}\text{Joules} \\ &= 1.692 \times 10^{-3}\text{eV} \end{aligned} \quad (10\text{–}18)$$

Solution to Problem 2: As time goes by... (Extra Credit)

Solution to Problem 2, part a.

Taking $p(x,t)$ as a joint ditribution on time and space, assuming th eindependence of both brings is equivalent to stating: $p(x,t) = F(t) \times \Phi(x)$

Solution to Problem 2, part b.

Wavefunctions are complex functions whose square is a probability distributions. Taking the assumption of independence stated in the previous part:

$$p(x,t) = F(t)\Phi(x) \quad (10\text{–}19)$$

$$= f(t)^\dagger f(t)\phi(x)^\dagger \phi(x) \quad (10\text{–}20)$$

the condition expressed in the problem statement is that the system is stationary, and consequently: $p(x,t) = p(x)$. This demands that $F(t) = 1$, that is $f(t)^\dagger f(t) = |f(t)|^2 = 1$

Solution to Problem 2, part c.

We rearranged Schrdinger equation after separating the wavefunction into functions of time and space by putting all the functions depending on time to the left and the functions depending on space to the right, and we got

$$i\hbar \frac{1}{f(t)} \left(\frac{df(t)}{dt} \right) = - \left(\frac{\hbar^2}{2m} \right) \frac{1}{\phi(x)} \left(\frac{d^2\phi(x)}{dx^2} \right) \quad (10\text{–}21)$$

What we just did is a common strategy to solve some partial differential equations, it is referred to as "separation of variables"; we assumed that the solution could be written as a product of two functions, each depending on a different variable (we separated the variables), and then rearranged the equation. The result is a function of time on the left hand side, and a function of space on the right hand side. In general the only way that two such functions could be equal is if they are both constants.

This argument is not necessarily transparent to the uninitiated. However, it is fairly straightforward, think of drawing a function in the paper and trying to make it equal to the reading of your watch, they better be both constant functions, i.e. both equal to a constant throughout their domain, the same constant. Appreciating this subtlety allows us to consider each domain (time and space) separately, and greatly simplifies calculations (the price we pay is generality, math rarely offers free lunches).

Se take the left hand side of equation 10–21, and equal it to a constant, let us call it C. The following derivation solves parts c to e

$$i\hbar \frac{1}{f(t)} \left(\frac{df(t)}{dt} \right) = C$$

pass $i\hbar$ to the right, and use the fact that $\frac{1}{i} = -i$

$$\hbar \frac{1}{f(t)} \left(\frac{df(t)}{dt} \right) = -i \frac{C}{\hbar}$$

Note that what remains at the left hand side is the derivative of a logarithm,

$$\frac{d\left(\ln f(t)\right)}{dt} = -i \frac{C}{\hbar}.$$

If you do the replacement $y(t) = \ln f(t)$,

$$\frac{dy}{dt} = -i \frac{C}{\hbar}$$

what you get is a simple derivative that can be integrated easily,

$$y(t) = \frac{-iCt}{\hbar}.$$

Undoing the change of variable,

$$\ln f(t) = \frac{-iCt}{\hbar}$$

and solving for f(t), yields the function f(t)

$$f(t) = e^{\frac{-iCt}{\hbar}}. \tag{10–22}$$

The procedure you just followed may seem cumbersome at first, but is worth taking the time to understand it because you will see it appear in a rich variety of different contexts.

Solution to Problem 2, part f.

To be mathematically rigorous we should prove that it division by $f(t)$ is possible in equation 10–21, that is, that the left hand side is always defined. To verify it we should examine the behavior in the limit $f(t) = 0$.

Solution to Problem 2, part g.

The difference between static and stationary is a subtle one. Some dynamic process is stationary if some relevant physical properties remain constant despite the fact that the process is dynamic. In this case, the probability distribution remains unperturbed, despite the fact that the wavefunction $f(t)$ changes over time. This means that the amplitude of the wavefunction remains constant and only its phase, otherwise unmeasurable, changes. Indeed, the fact that phase cannot be measured is used very often in quantum mechanics to prove very interesting results.