Name (1%): Your Name

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
Department of Mechanical Engineering

6.050J/2.110J **Information and Entropy** **Spring 2006**

Final Exam Solutions

Solution to Problem 1: Who's Who (14%)

Avogadro	adiabatic	Boltzmann	Boole	Carnot	intensive	Hamming
Huffman	entropy	Joule	continuous	Kraft	Lempel	Maxwell
Morse	Reed	Schrödinger	Shannon	Solomon	Welsh	Ziv
Boltzmann's Constant	Jaynes	ASCII	Bayes	GIF	discrete	Kelvin

Solution to Problem 1, part a.

Boltzmann's Constant or entropy can have units of Joules/Kelvin.

Solution to Problem 1, part b.

ASCII is a fixed length code.

Solution to Problem 1, part c.

The measure of how much two bit strings of equal length differ is named after Hamming.

Solution to Problem 1, part d.

While an MIT student in the 1940s, Kraft invented a famous inequality for his master's thesis.

Solution to Problem 1, part e.

Ziv, Lempel, Welsh was one of three engineers whose name is used for a lossless compression technique for the popular GIF image format.

Solution to Problem 1, part f.

The part of the heat-engine cycle without change in entropy is adiabatic.

Solution to Problem 1, part g.

The Boltzmann constant is approximately $1.38 \times 10^{-}23$ Joules per Kelvin.

Solution to Problem 1, part h.

The military engineer Carnot showed that all reversible heat engines have the same efficiency.

Solution to Problem 1, part i.

The channel capacity theorem proved by __Shannon__ states a possibility, not how to achieve it.

Solution to Problem 1, part j.

The algebra of binary numbers is named after the mathematician __Boole__, who had been a childhood prodigy in Latin, publishing at the age of 12.

Solution to Problem 1, part k.

__Maxwell__ conceived a Demon to show the statistical nature of the Second Law of Thermodynamics.

Solution to Problem 1, part l.

__Jaynes__ promoted the Principle of Maximum Entropy as an unbiased way of assigning probabilities.

Solution to Problem 1, part m.

A thermodynamic quantity that is the same for two systems in contact is __intensive__.

Solution to Problem 1, part n.

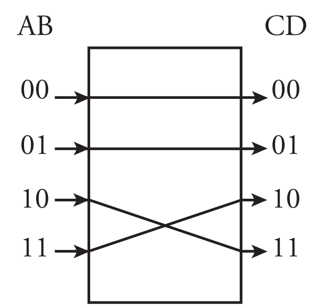
The energy values of an electron wavefunction in a square well are __discrete__, not __continuous__.

Solution to Problem 2: Under Control (5%)

Your company has just purchased a large number of $C-NOT$ (controlled-not) logic gates which are really just XOR (exclusive or) gates with an extra output. They have two inputs A and B and two outputs C and D . One output, C, is merely a copy of A, and the other output is B if $A = 0$ or $NOT B$ if $A = 1$. In other words, the output D is B possibly run through a NOT gate depending on the input A. Your boss wants you to design all your circuits using only $C-NOT$ gates, so the company can save the cost of maintaining different components in its inventory. You wonder about the properties of this gate.

You start off by modeling the gate in terms of its transition probabilities, and noting some basic properties.

You know that a gate is reversible if its input can be inferred exactly from its output. Since you know nothing about how the gate might be used, you assume the four possible input combinations are equally probable.



Solution to Problem 2, part a.

In the diagram at the right, show the transition probabilities.

Solution to Problem 2, part b.

What is the input information in bits? __2__

Solution to Problem 2, part c.

What is the output information in bits? __2__

Solution to Problem 2, part d.

What is the loss in bits? __0__

Solution to Problem 2, part e.

What is the noise in bits? 0

Solution to Problem 2, part f.

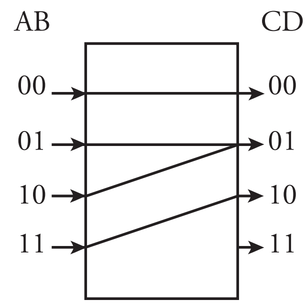
What is the mutual information in bits? 2

Solution to Problem 2, part g.

Is this gate reversible (yes or no)? yes

Solution to Problem 3: Out of Control (5%)

You have concluded that the $C-NOT$ gate from Problem 2 will not satisfy your needs, and look around for another. The incoming inspector tells you that some of the gates don't work right, and calls them $\mathbf{NOT-C-NOT}$ gates. They behave like $C-NOT$ gates except that they have a defective power supply which keeps the C output from being 1 when $D = 1$ even though the $C-NOT$ logic might call for them both to be 1 . This inspector thinks these gates are worthless but asks you want you think. You repeat your analysis for this gate, again assuming the four possible input combinations are equally probable.



Solution to Problem 3, part a.

In the diagram at the right, show the transition probabilities.

Solution to Problem 3, part b.

What is the input information in bits? 2

Solution to Problem 3, part c.

What is the output information in bits? 1.5

Solution to Problem 3, part d.

What is the loss in bits? 0.5

Solution to Problem 3, part e.

What is the noise in bits? 0

Solution to Problem 3, part f.

What is the mutual information in bits? 1.5

Solution to Problem 3, part g.

Is this gate reversible (yes or no)? No

Solution to Problem 4: Green Eggs and Hamming (15%)

Solution to Problem 4, part f.

How long a binary string (in bits) is needed to encode one order?

bits: 2

Solution to Problem 4, part f.

Give a suitable code using this number of bits.

Green Eggs: 00 Pancakes: 01 Greencake Combo: 10

Solution to Problem 4, part g.

What is the Hamming distance required between any pair of codewords to achieve single error-correction?

Minimum Hamming Distance: 3

Solution to Problem 4, part d.

A suitable code with this Hamming distance using five bits per order is:

Green Eggs: 00000 Pancakes: 00111 Greencake Combo: 11100

Solution to Problem 4, part g.

average bit length is less/equal/more: less

Solution to Problem 4, part f.

A Huffman code that takes advantage of these probabilities is:

Green Eggs: 00 Pancakes: 1 Greencake Combo: 01

Solution to Problem 4, part g.

The average code length of this code in bits is:

average length: 1.5

Solution to Problem 5: The Traveling SailMan (20%)

Solution to Problem 5, part a.

With this knowledge, what values of S, M, and L are possible?
We know that S and M can reach zero, since they are less than the average.

S_{min} 0 S_{max} 1/3 M_{min} 0 M_{max} 1/2 L_{min} 1/2 L_{max} 2/3

Solution to Problem 5, part b.

You decided to use the Principle of Maximum Entropy to estimate S, M, L, and the resulting uncertainty U about the number deployed. Express the entropy as a function of a single probability (any one of the three, S, M, or L).

$$\begin{aligned}
\text{Entropy}(S) &= S\log_2\left(\frac{1}{S}\right) + \frac{1-3S}{2}\log_2\left(\frac{2}{1-3S}\right) + \frac{S+1}{2}\log_2\left(\frac{2}{S+1}\right) \\
\text{Entropy}(M) &= \frac{1-2M}{3}\log_2\left(\frac{3}{1-2M}\right) + M\log_2\left(\frac{1}{M}\right) + \frac{2-M}{3}\log_2\left(\frac{3}{2-M}\right) \\
\text{Entropy}(L) &= (2L-1)\log_2\left(\frac{1}{2L-1}\right) + (2-3L)\log_2\left(\frac{1}{2-3L}\right) + L\log_2\left(\frac{1}{L}\right)
\end{aligned}$$

Solution to Problem 5, part c.

What probabilities S, M, and L did he want readers to infer, and what would be their resulting uncertainty in bits about the number of reconnaissance robots deployed in any one marina?

$S =$ __1/3__ $M =$ __0__ $L =$ __2/3__ Uncertainty = __0.92__

Solution to Problem 5, part d.

Compare this uncertainty to the value of U which you started to find earlier in this problem. Is this uncertainty

__X__ Less than U? ____ Equal to U? ____ Greater than U?

Solution to Problem 6: Variations on a Theme by Carnot (25%)

Solution to Problem 6, part a.

Does a heat engine traverse the rectangle in the above figure clockwise or counterclockwise?

CW or CCW? __CCW__

Solution to Problem 6, part b.

Give expressions for the cost, benefit and efficiency of the heat engine in terms of the quantities S_1, S_2, T_c, and T_h in the figure above.

$$\text{Cost} = Q_a = T_h(S_2 - S_1)$$

$$\text{Benefit} = Q_a - Q_b = (T_h - T_c)(S_2 - S_1)$$

$$\text{Efficiency} = \frac{\text{Benefit}}{\text{Cost}} = \frac{Q_a - Q_b}{Q_a} = \frac{T_h - T_c}{T_h}$$

Solution to Problem 6, part c.

Does a refrigerator traverse the rectangle in the above figure clockwise or counterclockwise?

CW or CCW? __CW__

Solution to Problem 6, part d.

Give expressions for the cost, benefit and coefficient of performance of the refrigerator in terms of the quantities S_1, S_2, T_c, and T_h in the figure above.

$$\text{Cost} = Q_d - Q_c = \underline{\quad (T_h - T_c)(S_2 - S_1) \quad}$$

$$\text{Benefit} = Q_c = \underline{\quad T_h(S_2 - S_1) \quad}$$

$$\text{Coefficient of Performance} = \frac{\text{Benefit}}{\text{Cost}} = \frac{Q_c}{Q_d - Q_c} = \underline{\quad \frac{T_h}{T_h - T_c} \quad}$$

Solution to Problem 6, part e.

Does a heat pump traverse the rectangle in the figure above clockwise or counterclockwise?

CW or CCW? CW

Solution to Problem 6, part f.

Give expressions for the cost, benefit and coefficient of performance of the heat pump in terms of the quantities S_1, S_2, T_c, and T_h in the figure above.

$$\text{Cost} = Q_f - Q_e = \underline{\quad (T_h - T_c)(S_2 - S_1) \quad}$$

$$\text{Benefit} = Q_f = \underline{\quad T_h(S_2 - S_1) \quad}$$

$$\text{Coefficient of Performance} = \frac{\text{Benefit}}{\text{Cost}} = \frac{Q_f}{Q_f - Q_e} = \underline{\quad \frac{T_h}{T_h - T_c} \quad}$$

Solution to Problem 7: Beating the Carnot Engine (15%)

Solution to Problem 7, part b.

$$\begin{aligned}
\alpha &= \ln\left(e^{H_j/T_j} + e^{-H_j/T_j}\right) \\
&= \ln(2\cosh\frac{H_j}{T_j}) \\
E_j &= e^{-\alpha}\left(-H_j e^{H_j/T_j} + H_j e^{-H_j/T_j}\right) \\
&= -H_j\left(\frac{e^{H_j/T_j} - e^{-H_j/T_j}}{e^{H_j/T_j} + e^{-H_j/T_j}}\right) \\
&= -H_j \tanh\frac{H_j}{T_j} \\
S_j &= \alpha + E_j/T_j \\
&= \ln(2\cosh\frac{H_j}{T_j}) - \frac{H_j}{T_j}\tanh\frac{H_j}{T_j}
\end{aligned}$$

$$\eta = \underline{\quad 1/3 \quad}$$

Solution to Problem 7, part b.

We already know that the Carnot engine gives the highest possible efficiency for an engine working between two reservoirs in constant tempratures. Moreover, in Ben's proposed cycle, the system actually interacts only with one reservoir and according to the the Kelvin-Planck statement of the second law of thermodynamics, it is not possible to receive heat from one reservoir and convert it completely to work, which is exactly what Ben is hoping for.

Solution to Problem 7, part c.

Just evaluate the expressions given in part (a):

$E_a =$ $-200 \times 0.75 = -150$

$E_b =$ $-300 \times 0.75 = -225$

$E_e =$ $-300 \times 0.9 = -270$

$S_a =$ $1.15 - 0.75 = 0.4$

$S_b =$ 0.4

$S_e =$ $1.55 - 1.5 \times 0.9 = 0.2$

Solution to Problem 7, part d.

$$w_{ab} = E_b - E_a = \text{-75} \quad q_{be} = E_e - E_b = \text{-45}$$

$$q_{be} = (S_a - S_e)T_a = 40 \quad w_{be} = E_a - E_e - q_{be} = 80$$

Solution to Problem 7, part e.

Apparently, the engine does not work at all. The net work done is +5 meaning that the engine does negative work. In fact, the engine receives $w = 5$ and converts it entirely to heat! No way to beat the Carnot engine!