Name (1%): ________________________________________

# Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science
Department of Mechanical Engineering

**6.050J/2.110J** **Information and Entropy** **Spring 2006**

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**Issued: May 23, 2006, 9:00 AM** **Final Exam** **Due: May 23, 2006, 12:00 AM**

**Note:** Please write your name at the top of each page in the space provided. The last page may be removed and used as a reference table for the calculation of logarithms.

## Problem 1: Who's Who (14%)

For each statement, fill in a name from the box that most closely matches. There are no repeated answers.

| | | | | | | |
|---|---|---|---|---|---|---|
| Avogadro | adiabatic | Boltzmann | Boole | Carnot | intensive | Hamming |
| Huffman | entropy | Joule | continuous | Kraft | Lempel | Maxwell |
| Morse | Reed | Schrödinger | Shannon | Solomon | Welsh | Ziv |
| Boltzmann's Constant | Jaynes | ASCII | Bayes | GIF | discrete | Kelvin |

a. ______________________ can have units of Joules/Kelvin.

b. ______________________ is a fixed length code.

c. The measure of how much two bit strings of equal length differ is named after __________.

d. While an MIT student in the 1940s, __________ invented a famous inequality for his master's thesis.

e. __________ was one of three engineers whose name is used for a lossless compression technique for the popular GIF image format.

f. The part of the heat-engine cycle without change in entropy is ______________________.

g. The __________ constant is approximately $1.38 \times 10^{-}23$ Joules per Kelvin.

h. The military engineer __________ showed that all reversible heat engines have the same efficiency.

i. The channel capacity theorem proved by __________ states a possibility, not how to achieve it.

j. The algebra of binary numbers is named after the mathematician __________, who had been a childhood prodigy in Latin, publishing at the age of 12.

k. __________ conceived a Demon to show the statistical nature of the Second Law of Thermodynamics.

l. __________ promoted the Principle of Maximum Entropy as an unbiased way of assigning probabilities.

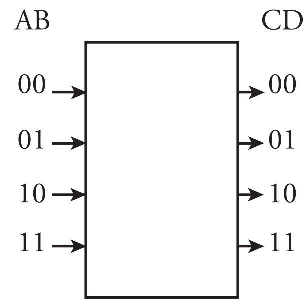
m. A thermodynamic quantity that is the same for two systems in contact is ______________________.

n. The energy values of an electron wavefunction in a square well are ______________________, not ______________________.

## Problem 2: Under Control (5%)

Your company has just purchased a large number of $C-NOT$ (controlled-not) logic gates which are really just $XOR$ (exclusive or) gates with an extra output. They have two inputs $A$ and $B$ and two outputs $C$ and $D$ . One output, $C$, is merely a copy of $A$, and the other output is $B$ if $A=0$ or $NOTB$ if $A=1$. In other words, the output $D$ is $B$ possibly run through a $NOT$ gate depending on the input $A$. Your boss wants you to design all your circuits using only $C-NOT$ gates, so the company can save the cost of maintaining different components in its inventory. You wonder about the properties of this gate.

You start off by modeling the gate in terms of its transition probabilities, and noting some basic properties.




You know that a gate is reversible if its input can be inferred exactly from its output. Since you know nothing about how the gate might be used, you assume the four possible input combinations are equally probable.

a. In the diagram at the right, show the transition probabilities.

b. What is the input information in bits? ______________________

c. What is the output information in bits? ______________________

d. What is the loss in bits? ______________________

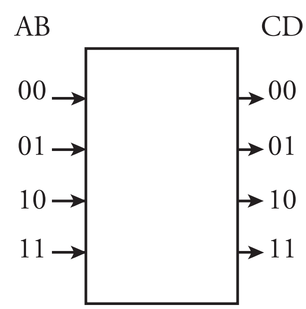
e. What is the noise in bits? ______________________

f. What is the mutual information in bits? ______________________

g. Is this gate reversible (yes or no)? ______________________

## Problem 3: Out of Control (5%)

You have concluded that the $C-NOT$ gate from Problem 2 will not satisfy your needs, and look around for another. The incoming inspector tells you that some of the gates don't work right, and calls them $\mathbf{NOT-C-NOT}$ gates. They behave like $C-NOT$ gates except that they have a defective power supply which keeps the $C$ output from being 1 when $D=1$ even though the $C-NOT$ logic might call for them both to be 1 . This inspector thinks these gates are worthless but asks you want you think. You repeat your analysis for this gate, again assuming the four possible input combinations are equally probable.




a. In the diagram at the right, show the transition probabilities.

b. What is the input information in bits? ______________________

c. What is the output information in bits? ______________________

d. What is the loss in bits? ______________________

e. What is the noise in bits? ______________________

f. What is the mutual information in bits? ______________________

g. Is this gate reversible (yes or no)? ______________________

## Problem 4: Green Eggs and Hamming (15%)

The new MIT campus hotspot, Buzz, has a new breakfast menu with three items: Green Eggs, Pancakes or the Greencake Combo (Green Eggs and Pancakes together). Since this is MIT, customers are required to place their orders in binary strings.

a. How long a binary string (in bits) is needed to encode one order?

bits: ____________________________

b. Give a suitable code using this number of bits.

Green Eggs: ______________ Pancakes: ______________ Greencake Combo: ______________

c. The chef has noticed that several orders have been received incorrectly, because of the high ambient noise level. She wants you to design a new code for the three orders that will allow all orders to be understood even if one of the bits is wrong. From your experience with information and entropy, you realize that what she wants is a code that will allow single-error correction. What is the Hamming distance required between any pair of codewords to achieve this?

Minimum Hamming Distance: ______________

d. Give a suitable code with this Hamming distance using five bits per order.

Green Eggs: ______________ Pancakes: ______________ Greencake Combo: ______________

e. The chef also has to transmit all the orders, in the order they are received, to the stock room. She has a noise-free channel for this purpose, but it is very expensive and she wants to encode the orders in the least number of bits. You recommend a Huffman code because they were invented at MIT. Do you expect the average Huffman code length to be less than, equal to, or more than your answer to part (a)?

average bit length is less/equal/more: ______________

f. Experience has shown that students order Green Eggs 30% of the time, Pancakes 50% of the time, and the Greencake combo 20% of the time. Give a Huffman code that takes advantage of these probabilities.

Green Eggs: ______________ Pancakes: ______________ Greencake Combo: ______________

g. What is the average code length of this code in bits?

average length: ______________

## Problem 5: The Traveling SailMan (20%)

Nothing annoys owners of expensive sailboats more than dirty sails. The SailMan robot solves this problem. Every night it swims from boat to boat, climbs aboard, and cleans the sails. This robot is helped by reconnaissance robots that report the location of boats with dirty sails to a central computer, which then plans the night's activities by solving the "Traveling SailMan" problem. Because of software limitations, reconnaissance robots can only be deployed in groups of 2 (Small), 4 (Medium), or 8 (Large).

Visiting a marina one evening, you noticed the SailMan robot and naturally wondered how many reconnaissance robots were deployed there. You expressed your knowledge in terms of probabilities (call them $S$, $M$, and $L$) that there were 2, 4, or 8 reconnaissance robots. You remembered from the SailMan Annual Report that the average number of reconnaissance robots deployed in a marina was 6.

a. With this knowledge, what values of $S$, $M$, and $L$ are possible?

$S_{min}$________ $S_{max}$________ $M_{min}$________ $M_{max}$________ $L_{min}$________ $L_{max}$________

b. You decided to use the Principle of Maximum Entropy to estimate $S$, $M$, $L$, and the resulting uncertainty $U$ about the number deployed. Express the entropy as a function of a single probability (any one of the three, $S$, $M$, or $L$).

Entropy = ________________________________________________

Before you had a chance to evaluate $U$, you happened to run into the the Fiscal Officer of SailMan, Inc., who said that when he wrote the Annual Report he wanted readers to think the high-priced configuration of 8 reconnaissance robots was selling better than it really was. He reported the average number of reconnaissance robots (6) correctly but then mentioned the largest value of $L$ that was consistent with this average, without actually saying it was correct (linguists would call this an "intended inference").

c. What probabilities $S$, $M$, and $L$ did he want readers to infer, and what would be their resulting uncertainty in bits about the number of reconnaissance robots deployed in any one marina?

$S$ = __________ $M$ = __________ $L$ = __________ Uncertainty = __________

d. Compare this uncertainty to the value of $U$ which you started to find earlier in this problem. Is this uncertainty

_______ Less than $U$? _______ Equal to $U$? _______ Greater than $U$?

## Problem 6: Variations on a Theme by Carnot (25%)

The ideal (and most efficient) version of a magnetic heat engine operates reversibly in a cycle that can be represented as a rectangle in the $T-S$ plane. The same physical device can also be operated reversibly as a refrigerator or as a heat pump to accomplish different goals. This problem asks you to quantify the effectiveness of these three systems in terms of the variables in the diagram below. The entropies and temperatures in the plot and the energies given in the description of each problem are all positive.

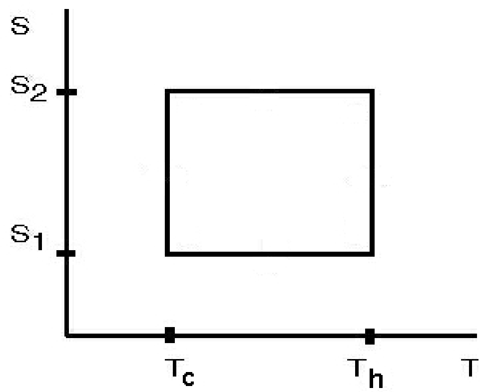



Figure 1: $T-S$ Cycle for a reversible heat engine

**Heat Engine:** The purpose of a heat engine is to convert heat to work. In each cycle a heat engine reversibly extracts $Q_a$ Joules of heat from a hot reservoir at temperature $T_h$, dumps a portion $Q_b$ Joules of that heat into a cooler environment at temperature $T_c$ and converts the remaining energy into $(Q_a - Q_b)$ Joules of useful work. The cost per cycle is the energy $Q_a$ taken from the hot reservoir and the benefit per cycle is the work $(Q_a - Q_b)$ performed.

a. Does a heat engine traverse the rectangle in the above figure clockwise or counterclockwise?

CW or CCW? ______________________

b. Give expressions for the cost, benefit and efficiency of the heat engine in terms of the quantities $S_1$, $S_2$, $T_c$, and $T_h$ in the figure above.

Cost $= Q_a =$ ______________________

Benefit $= Q_a - Q_b =$ ______________________

Efficiency $= \frac{\text{Benefit}}{\text{Cost}} = \frac{Q_a - Q_b}{Q_a} =$ ______________________

Continued on next page...

**Refrigerator:** The purpose of a refrigerator is to cool something below the temperature of its environment. Though the temperature of the object to be cooled will drop over time, the ideal model represents the cooled object as a constant-temperature object at a temperature $T_c$. In each cycle a refrigerator reversibly extracts $Q_c$ Joules of heat from a cold object, ejects a larger quantity $Q_d$ Joules of heat into a warmer environment at temperature $T_h$, and requires $(Q_d - Q_c)$ Joules of work to traverse the cycle. The cost per cycle is the $(Q_d - Q_c)$ Joules of work that must be performed and the benefit per cycle is the $Q_c$ Joules of heat taken from the cold object.

c. Does a refrigerator traverse the rectangle in the above figure clockwise or counterclockwise?

CW or CCW? ____________________

d. Give expressions for the cost, benefit and coefficient of performance of the refrigerator in terms of the quantities $S_1$, $S_2$, $T_c$, and $T_h$ in the figure above.

Cost $= Q_d - Q_c =$ ____________________

Benefit $= Q_c =$ ____________________

Coefficient of Performance $= \frac{\text{Benefit}}{\text{Cost}} = \frac{Q_c}{Q_d - Q_c} =$ ____________________

**Heat Pump:** The purpose of a heat pump is to warm something (e.g., your house) efficiently by making use of heat that is available at a lower temperature (e.g., an underground well on your property.) Though the temperature of your house will rise over time, the ideal model represents it as a constant-temperature reservoir at a temperature $T_h$. In each cycle a heat pump reversibly extracts $Q_e$ Joules of heat from a cold reservoir at temperature $T_c$, ejects a larger quantity $Q_f$ Joules of heat into a warmer environment and requires $(Q_f - Q_e)$ Joules of work to traverse the cycle. The cost per cycle is the $(Q_f - Q_e)$ Joules of work that must be performed and the benefit per cycle is the $Q_f$ Joules of heat added to the hot reservoir.

e. Does a heat pump traverse the rectangle in the figure above clockwise or counterclockwise?

CW or CCW? ____________________

f. Give expressions for the cost, benefit and coefficient of performance of the heat pump in terms of the quantities $S_1$, $S_2$, $T_c$, and $T_h$ in the figure above.

Cost $= Q_f - Q_e =$ ____________________

Benefit $= Q_f =$ ____________________

Coefficient of Performance $= \frac{\text{Benefit}}{\text{Cost}} = \frac{Q_f}{Q_f - Q_e} =$ ____________________

## Problem 7: Beating the Carnot Engine (15%)

Some alumni of 6.050 have established CheapEng Inc. where they develop heat engines based on the single dipole-magnetic field thermodynamic cycle that we have had in the course. They have hired another MIT graduate, Ben, who hasn't taken the course but thinks he knows enough to be able to improve the system. The system can be described as illustrated in figure 1 and has the following parameters:

| Point | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $H$ (A/m) | 200 | 300 | 150 | 100 |
| $T$ (K) | 200 | 300 | 300 | 200 |

Table 1: The Parameters of the system

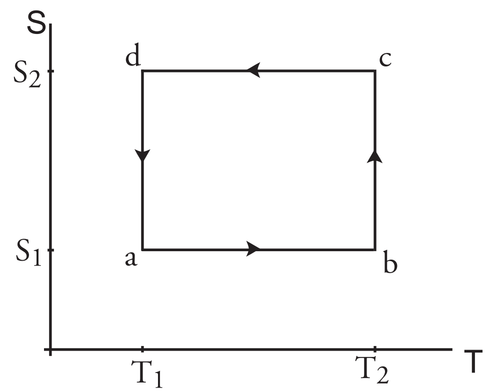



Figure 1: Heat Engine Cycle

where $T$ and $H$, represent the temperature and magnetic field, respectively.

a. Find the efficiency of the Carnot engine and show that the following expressions give the energy and entropy of each point $j$ in the equilibrium. Throughout this problem consider **the Boltzmann's constant $\mathbf{k_B = 1}$ (J/K) and the magnetic dipole moment $\mathbf{m = 1}$ (J · m/A)**.

$$E_j = -H_j \tanh \frac{H_j}{T_j}$$

$$S_j = \ln(2 \cosh \frac{H_j}{T_j}) - \frac{H_j}{T_j} \tanh \frac{H_j}{T_j}$$

$\eta =$ ____________________

| Point | $a$ | $b$ | $e$ |
|---|---|---|---|
| $H$ (A/m) | 200 | 300 | 300 |
| $T$ (K) | 200 | 300 | 200 |

Table 2: The Parameters of the new system.

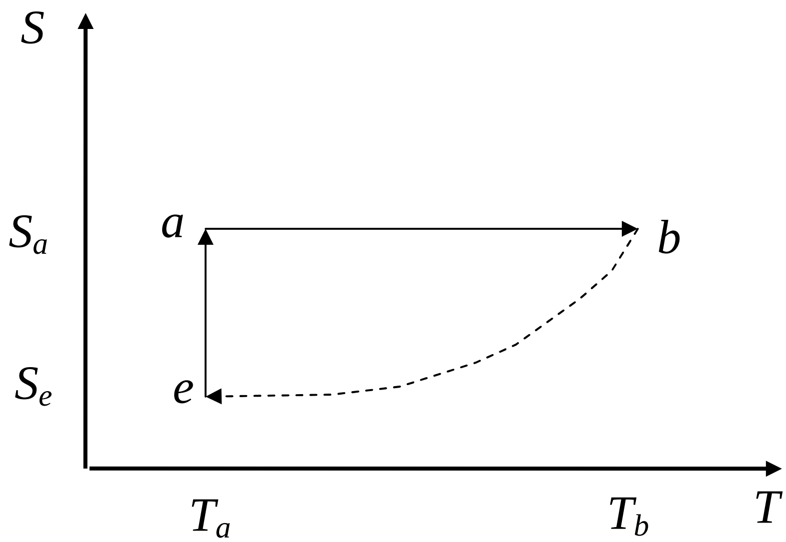



Figure 2: Ben's proposed cycle for achieving a higher efficiency.

Ben realizes that the system gives us positive work during the adiabatic process $a-b$ and net positive heat during its isothermal interaction with the reservoir 1, but then in the rest of the process does some unfavorable things like receiving energy from the reservoir 2 in $b-c$ and receiving work in reducing $H$ during $c-d$. He thinks that in order to increase the efficiency $\eta = w/Q_{received}$, he can get around these steps by performing the following irreversible process: he starts the cycle in $a$ and follows the same leg $a-b$. But, while the system is at temperature $T_2$ after the leg $a-b$, he promptly puts the system again in contact with the first reservoir and lets the system come to equilibrium $e$. Then, in an isothermal process, he reduces the magnetic field $H$ while system is still in contact with the first reservoir until the system reaches the initial point $a$. Figure 2 shows Ben's new cycle. Interestingly enough, Ben realizes that this completes the cycle and he does not need to perform any other process.

b. Based on the basic rules you have learned, do you think Ben's cycle will give a higher efficiency or not? Explain the reasons you have for your answer.

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c. Find the energy and entropy of the equilibrium points $a$, $b$ and $e$. You can use the approximate values presented in the last page.

$E_a =?$_______ $S_a =?$_______ $E_b =?$_______ $S_b =?$_______ $E_e =?$_______ $S_e =?$_______

d. Find the work and heat exchanged throughout the cycle.

$$w_{ab} = E_b - E_a =?\_\_\_\_\_\_\_ \quad q_{be} = E_e - E_b =?\_\_\_\_\_\_\_$$

$$q_{be} = (S_a - S_e)T_a =?\_\_\_\_\_\_\_ \quad w_{be} = E_a - E_e - q_{be} =?\_\_\_\_\_\_\_$$

e. Based on the results of the previous parts, try to justify your answer to part (b). Be as precise as possible in interpreting the above numbers.

## Logarithm and Entropy Table

This page is provided so that you may rip it off the exam to use as a separate reference table. In Table 3, the entropy $S = p\log_2(1/p) + (1-p)\log_2(1/(1-p))$.

| $p$ | 1/8 | 1/5 | 1/4 | 3/10 | 1/3 | 3/8 | 2/5 | 1/2 | 3/5 | 5/8 | 2/3 | 7/10 | 3/4 | 4/5 | 7/8 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\log_2(1/p)$ | 3.00 | 2.32 | 2.00 | 1.74 | 1.58 | 1.42 | 1.32 | 1.00 | 0.74 | 0.68 | 0.58 | 0.51 | 0.42 | 0.32 | 0.18 |
| $S$ | 0.54 | 0.72 | 0.81 | 0.88 | 0.92 | 0.95 | 0.97 | 1.00 | 0.97 | 0.95 | 0.92 | 0.88 | 0.81 | 0.72 | 0.54 |

Table 3: Table of logarithms in base 2 and entropy in bits

$$\tanh(1) \approx 0.75 \qquad \tanh(1.5) \approx 0.9$$
$$\ln(2\cosh(1)) \approx 1.15 \qquad \ln(2\cosh(1.5)) \approx 1.55$$