Name (1%): ____

Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

Department of Mechanical Engineering

6.050J/2.110J **Information and Entropy** **Spring 2005**

Issued: May 20, 2005, 1:30 PM **Final Exam** **Due: May 20, 2005, 4:30 PM**

Note: Please write your name at the top of each page in the space provided. The last page may be removed and used as a reference table for the calculation of logarithms.

Problem 1: Who's Who (14%)

For each statement, fill in a name from the box that most closely matches. There are no repeated answers.

Avogadro	Bayes	Boltzmann	Boole	Carnot	Gibbs	Hamming
Huffman	Jaynes	Joule	Kelvin	Kraft	Lempel	Maxwell
Morse	Reed	Schrödinger	Shannon	Solomon	Welsh	Ziv

a. ____________, a 19-th century physicist, has a unit of temperature named after him.

b. Reportedly ____________ secluded himself in his mountain cabin, with pearls in his ears to muffle the sound, and his girlfriend in his bed to inspire him, and came up with an equation to calculate the wave function of a quantum system.

c. The measure of how much two bit strings of equal length differ is named after ____________.

d. While an MIT student in the 1940s, ____________ invented a famous inequality for his master's thesis.

e. ____________ was one of three engineers whose name is used for a lossless compression technique for the popular GIF image format.

f. A famous inequality is named after ____________, who received the first doctorate in engineering in America, and later was on the faculty at Yale University.

g. The ____________ constant is approximately $1.38 \times 10^{-}23$ Joules per Kelvin.

h. The military engineer ____________ showed that all reversible heat engines have the same efficiency.

i. The channel capacity theorem proved by ____________ states a possibility, not how to achieve it.

j. The algebra of binary numbers is named after the mathematician ____________, who had been a childhood prodigy in Latin, publishing at the age of 12.

k. ____________ conceived a Demon to show the statistical nature of the Second Law of Thermodynamics.

l. ____________ promoted the Principle of Maximum Entropy as an unbiased way of assigning probabilities.

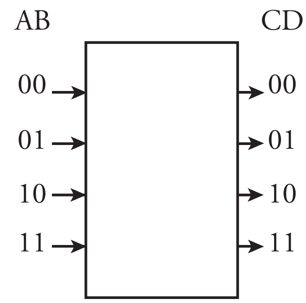
m. During an ocean journey ____________ heard that electricity could be transmitted instantaneously, and in a fit of creativity invented a code to use it to transmit arbitrary information.

n. Over one weekend ____________ solved a problem posed in a problem set for an MIT graduate subject, thereby inventing the shortest possible variable length code.

Problem 2: Under Control (5%)

Your company has just purchased a large number of $C - NOT$ (controlled-not) logic gates which are really just XOR (exclusive or) gates with an extra output. They have two inputs A and B and two outputs C and D . One output, C, is merely a copy of A, and the other output is B if $A = 0$ or $NOTB$ if $A = 1$. In other words, the output D is B possibly run through a NOT gate depending on the input A. Your boss wants you to design all your circuits using only $C - NOT$ gates, so the company can save the cost of maintaining different components in its inventory. You wonder about the properties of this gate.

You start off by modeling the gate in terms of its transition probabilities, and noting some basic properties.



You know that a gate is reversible if its input can be inferred exactly from its output. Since you know nothing about how the gate might be used, you assume the four possible input combinations are equally probable.

a. In the diagram at the right, show the transition probabilities.

b. What is the input information in bits? ______________________

c. What is the output information in bits? ______________________

d. What is the loss in bits? ______________________

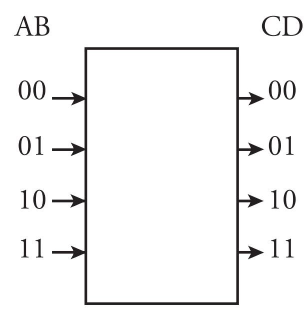
e. What is the noise in bits? ______________________

f. What is the mutual information in bits? ______________________

g. Is this gate reversible (yes or no)? ______________________

Problem 3: Out of Control (5%)

You have concluded that the $C - NOT$ gate from Problem 2 will not satisfy your needs, and look around for another. The incoming inspector tells you that some of the gates don't work right, and calls them **NOT − C − NOT** gates. They behave like $C - NOT$ gates except that they have a defective power supply which keeps the C output from being 1 when $D = 1$ even though the $C - NOT$ logic might call for them both to be 1 . This inspector thinks these gates are worthless but asks you want you think. You repeat your analysis for this gate, again assuming the four possible input combinations are equally probable.



a. In the diagram at the right, show the transition probabilities.

b. What is the input information in bits? ______________________

c. What is the output information in bits? ______________________

d. What is the loss in bits? ______________________

e. What is the noise in bits? ______________________

f. What is the mutual information in bits? ______________________

g. Is this gate reversible (yes or no)? ______________________

Problem 4: MIT Customer Complaint Department (15%)

You have recently been elected President of the UA, and it is your job to transmit student complaints to MIT President Susan Hockfield so they (hopefully) can be addressed rapidly. According to the UA's research, all student complaints fall into one of six categories, with percentages shown:

%	Complaint
50%	Not enough homework
30%	Campus dining options too diverse
10%	Tuition too low
5%	Administration too attentive
5%	Classes too easy

Unfortunately, Hockfield doesn't have much time, so she instructs you to only send very short messages to her. Because you've taken 6.050 you know about coding schemes, so you decide to encode the complaints above in a Huffman code.

a. Design a Huffman code for the complaints above.

Complaint	Code
Not enough homework	______________
Campus dining options too diverse	______________
Tuition too low	______________
Administration too attentive	______________
Classes too easy	______________

b. What is the average number of bits to send one complaint?

Average # of bits/complaint: ____________________________

Problem 5: Zero Power Cooler (30%)

Having landed a high-paying job in Boston, you decide to build a new house for yourself. You vow to use only energy-efficient techniques, based on what you know about information and entropy. Of course your house needs heating in the winter, cooling in the summer, and beverage refrigeration all year long. You decide to design a device that cools the beverage storage room without requiring any work and therefore no electricity bill.

Of course you are familiar with the Carnot efficiency results for reversible heat exchangers. You know that ordinarily heat engines, heat pumps, and refrigerators operate between two reservoirs at two different temperatures, and either produce or require work. But in your case you have reservoirs at three different temperatures (storage room, house, and outside). You wonder whether a reversible heat exchange system could take heat out of the storage room and either put heat into or take heat from the other two reservoirs, with no work.

a. First consider the winter. Model the outside environment as a large heat reservoir at $275K$ (about 2^oC or 35^oF). You want your house to be at $295K$ (about 22^oC or 72^oF) and the storage room at $280K$ (about 7^oC or 44^oF).

   Is such a system possible, at least in principle? Or would it violate the Second Law of Thermodynamics? ______________________

   If it is possible, give the heat that would be exchanged with each of the three reservoirs if 1 Joule of heat is taken from the storage room at $280K$. If it is not possible, briefly explain why.

b. Next, consider the summer, when the outside temperature is $300K$ (about 27^oC or 80^oF). The house and storage room temperatures are the same as for the winter, namely $295K$ and $280K$. You wonder whether a reversible heat exchange system can be designed to take energy out of the storage area without requiring any work.

   Is such a system possible, at least in principle? Or would it violate the Second Law of Thermodynamics? ______________________

   If it is possible, give the heat that would be exchanged with each of the three reservoirs if 1 Joule of heat is taken from the refrigerator at $280K$. It is is not possible, briefly explain why.

Problem 6: Casino (20%)

On your vacation you find yourself in a gambling casino that caters to nerds. One of the games catches your attention because you wonder about the probabilities. In this game, you pay a nickel (5 cents) to play, and then a coin is chosen at random from a big jar and given to you. The jar contains pennies (worth 1 cent), nickels (5 cents), and quarters (25 cents). This game, unlike slot machines, has some payout each time you play. You ask the owner of the casino about the probabilities P, N, and Q of the three coins (penny, nickel, and quarter) being selected, but the only information he gives you is that the average payout is 4 cents per play. You notice that the jar is so large that P, N, and Q remain the same all day long.

a. What values of P, N, and Q (nonnegative, no larger than 1) are possible?

$\underset{P_{\text{Min}}}{\underline{\hspace{2cm}}} \le P \le \underset{P_{\text{Max}}}{\underline{\hspace{2cm}}}$ $\qquad$ $\underset{N_{\text{Min}}}{\underline{\hspace{2cm}}} \le N \le \underset{N_{\text{Max}}}{\underline{\hspace{2cm}}}$ $\qquad$ $\underset{Q_{\text{Min}}}{\underline{\hspace{2cm}}} \le Q \le \underset{Q_{\text{Max}}}{\underline{\hspace{2cm}}}$

b. You decide to estimate P, N, and Q using the Principle of Maximum Entropy (if the owner made this choice it would be exciting to nerd customers, because they would gain the most information when learning which coin is chosen). (b) Start this estimate by eliminating P and N and writing the entropy (your uncertainty about the coin to be selected) as a function of Q:

Entropy = ____

Without a calculator you can't find the maximum of this expression, so instead you guess that it is about 0.5 bits.

Then you realize that the owner need not stock the jar using maximum entropy. He could set the probabilities so as to maximize the number of quarters paid out, while still keeping an average payout of 4 cents (this choice would be exciting to non-nerd customers hoping to strike it rich).

c. With this strategy, what are P, N, Q, and the entropy (to two decimal places)?

P = __________, N = __________, Q = __________, Entropy = ______________.

d. Was your earlier quick guess for the entropy (when you thought the Principle of Maximum Entropy was used by the owner) a good one? Explain your answer.

____

____

____

____

e. **(Extra credit)** Another strategy for the owner would be to stock the jar with no quarters at all, so Q = 0 (this choice would appeal only to nerd customers). For this strategy, what are P, N, Q, and the entropy (to two decimal places)?

P = __________, N = __________, Q = __________, Entropy = ______________.

Problem 7: The Telephone Game (30%)

The "Telephone Game" illustrates how correct information gets converted into false rumors. In the game one person (Alice) sends a message to another (Bob) through a chain of humans, each of whom potentially corrupts the message. Thus Alice tells person #1, who then tells person #2, and so on until the last person in the chain tells Bob. Then Bob and Alice announce their versions of the message, normally accompanied by amazement at how different they are.

Consider the case where a single bit is being passed and each person in the chain has a 20% probability of passing on a bit different from the one she received. Thus we model each person in the chain as a symmetric binary channel as shown in Figure 1.

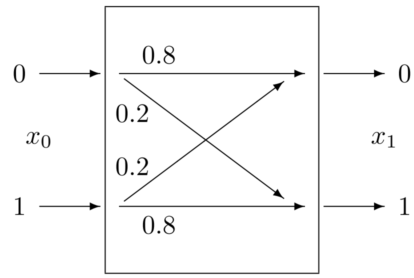


Figure 1: Simple model of a person

The game is being demonstrated for you at a party one day. Alice and Bob take their positions at opposite ends of the chain, and Alice whispers the value of x_0 to person #1. You know that person #1, like the other members of the chain, has a 20% probability of changing the bit she hears. Parts *a.* and *b.* concern the model for this person, and parts *c.* and *d.* concern the behavior of the chain.

a. At first, you do not know what Alice has told person #1. Naturally, you express your state of knowledge in terms of the two probabilities $p(x_0 = 0)$ and $p(x_0 = 1)$. To avoid any unintended bias you use the Principle of Maximum Entropy to conclude that each of these probabilities is equal to 0.5. Then you calculate your uncertainty I_0 about the value of x_0, the output probabilities $p(x_1 = 0)$ and $p(x_1 = 1)$, and your uncertainty I_1 about the value of x_1. Then you calculate the channel noise N, channel loss L, and mutual information M, all in bits:

$p(x_0 = 0) =$ ____0.5____ $p(x_0 = 1) =$ ____0.5____ $I_0 =$ ____________ bits

$p(x_1 = 0) =$ ________ $p(x_1 = 1) =$ ________ $I_1 =$ ____________ bits

$N =$ ____________ $L =$ ____________ $M =$ ____________

Next, consider the behavior of a cascade of independent identical channels representing the individuals passing the message, as illustrated in Figure 2. Let x_k represent the output of the k-th channel.

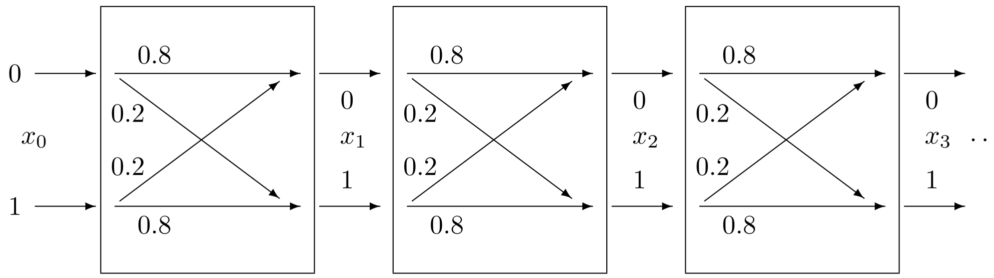


Figure 2: Simple model of the telephone game

b. Now suppose that you know that Alice's bit is 0, so that $x_0 = 0$. Having just calculated probabilities for x_1, you wonder how much you know about the other values, x_2, x_3, x_4, $\ldots x_k \ldots$. Is it true that $p(x_k = 0) > 0.5$ for every channel output x_k? In other words, is every value passed along more likely to be 0 than 1? Write a paragraph defending your conclusion. If possible, make this an outline of a proof. You may find some version of the principle of induction helpful. (Pictures and equations are allowed in the paragraph, if needed.)

c. You conclude (correctly) that $p(x_k = 0)$ decreases as the message moves along the chain, i.e.,

$$p(x_0 = 0) > p(x_1 = 0) > p(x_2 = 0) > \ldots > p(x_k = 0) > p(x_{k+1} = 0) > \ldots \tag{1}$$

Let I_k be your uncertainty about x_k in bits. Does the uncertainty about x_k increase as you move through successive channels? In other words, is the following sequence of inequalities true?

$$I_0 < I_1 < I_2 < \ldots < I_k < I_{k+1} < \ldots \tag{2}$$

Write a paragraph defending your answer.

Logarithm and Entropy Table

This page is provided so that you may rip it off the exam to use as a separate reference table. In Table 1, the entropy $S = p\log_2(1/p) + (1-p)\log_2(1/(1-p))$.

p	1/8	1/5	1/4	3/10	1/3	3/8	2/5	1/2	3/5	5/8	2/3	7/10	3/4	4/5	7/8
$\log_2(1/p)$	3.00	2.32	2.00	1.74	1.58	1.42	1.32	1.00	0.74	0.68	0.58	0.51	0.42	0.32	0.18
S	0.54	0.72	0.81	0.88	0.92	0.95	0.97	1.00	0.97	0.95	0.92	0.88	0.81	0.72	0.54

Table 1: Table of logarithms in base 2 and entropy in bits