Massachusetts Institute of Technology
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6.050J/2.110J **Information and Entropy** **Spring 2004**

Problem Set 8 Solutions

Solution to Problem 1: Uncertain Employment

Solution to Problem 1, part a.

i. We will need to rearrange Equations 8–1 and 8–2. First we get the following from 8–2

$$p(S) = 1.8 - 2p(E) \tag{8–3}$$

and substituting this into 8–1 we have

$$\begin{aligned} 1 &= p(E) + p(S) + p(U) \\ &= p(E) + 1.8 - 2p(E) + p(U) \\ &= 1.8 - p(E) + p(U) \qquad (8–4) \end{aligned}$$

$$p(E) = 0.8 + p(U) \tag{8–5}$$

Once $p(E)$ and $p(U)$ are determined, $p(A)$ is determined also:

$$\begin{aligned} p(S) &= 0.2 - 2p(U) \\ &= 1.8 - 2p(E) \qquad (8–6) \\ p(U) &= p(E) - 0.8 \\ &= 0.1 - 0.5p(S) \qquad (8–7) \\ p(E) &= 0.8 + p(U) \\ &= 0.9 - 0.5p(S) \qquad (8–8) \end{aligned}$$

And so, since both $p(U)$, $p(S)$, and $p(E)$ must be between 0 and 1, we see that $p(E)$ must be between 0.8 and 0.9. Furthermore, $p(U)$ can only be between 0 and 0.1. These are reasonable answers: if the average employment is to be kept high, then we would expect that fully-employed rate would be high and unemployment rate would be low.

ii. The equation for the entropy is as follows:

$$\begin{aligned} H &= p(E)\log_2\left(\frac{1}{p(E)}\right) + p(S)\log_2\left(\frac{1}{p(S)}\right) + p(U)\log_2\left(\frac{1}{p(U)}\right) \\ &= p(E)\log_2\left(\frac{1}{p(E)}\right) + (1.8 - 2p(E))\log_2\left(\frac{1}{1.8 - 2p(E)}\right) + \\ &\quad (p(E) - 0.8)\log_2\left(\frac{1}{p(E) - 0.8}\right) \qquad (8–9) \end{aligned}$$

Plotting this for $p(F) = 0.4$ to 0.9 we get the graph shown in Figure 8–1

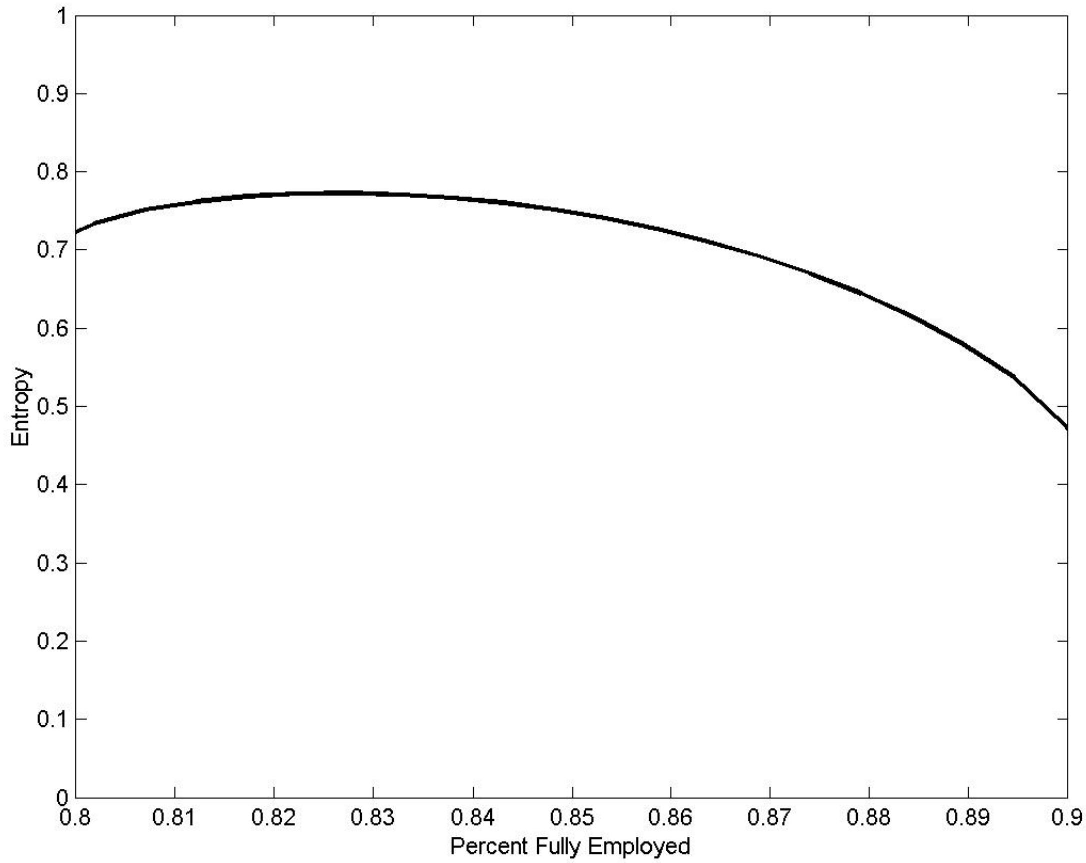

Figure 8–1: Entropy of the employment probability distribution

iii. The maximum entropy of $H = 0.7724$ bits is at $p(E) = 0.8242$, which gives values of $p(S) = 0.1516$ and $p(U) = 0.0242$.

Solution to Problem 1, part b.

The entropy should be less than (a-iii) because, after all, that value was calculated with a procedure known as the Principle of Maximum Entropy. The maximum value of $p(E)$ consistent with the constraints is 0.90.

Solution to Problem 1, part c.

If $p(E)$ is 0.90, then $p(S) = 0$ and $p(U) = 0.1$. The entropy at this point is $H = 0.4713$ bits.

Solution to Problem 1, part d.

If $p(U)$ is set at its minimum level of 0, then $p(E) = 0.8$ and $p(S) = 0.2$ The entropy at this point is, reading from the graph, 0.723.

Solution to Problem 1, part e.

The minimum entropy is zero. Any distribution that puts full weight into one of the variables will achieve zero entropy.

Solution to Problem 2: Candy Raffle

Solution to Problem 2, part a.

We know that the contents of the jar have a mass of 2400 grams. The equation that governs this is

$$2.5n(T) + 1.4n(K) + 0.8n(M) = 2400 \tag{8–10}$$

The numbers compatible with this equation are calculated by dividing the weight of each piece of candy into the maximum weight of the contents:

$$\begin{aligned} n(T) &= 0 \text{ to } 960 \\ n(K) &= 0 \text{ to } 1714 \\ n(M) &= 0 \text{ to } 3000 \end{aligned} \tag{8–11}$$

Solution to Problem 2, part b.

Knowing there are 1000 pieces of candy in the jar adds a second constraint to our system.

$$n(T) + n(K) + n(M) = 1000 \tag{8–12}$$

To calculate the percentages that maximize the uncertainty, we will cast our constraints as follows:

$$\begin{aligned} n(M) &= 1000 - n(T) - n(K) \\ 2400 &= 2.5n(T) + 1.4n(K) + 800 - 0.8n(T) - 0.8n(K) \\ 2400 &= 1.7n(T) + 0.6n(K) + 800 \end{aligned} \tag{8–13}$$

and thus, in terms of Tootsie Rolls:

$$\begin{aligned} n(K) &= 2666.66 - 2.83n(T) \\ n(M) &= 1.83n(T) - 1666.66 \end{aligned} \tag{8–14}$$

Since $n(K)$ cannot be less than zero, $n(T)$ cannot be greater than 2666.66/2.83, or 942 (rounding to the nearest integer). Furthermore, $n(T)$ cannot be less than 910, because this would then make $n(M)$ less than zero. So we have bounds on $n(T)$.

$$\begin{aligned} H &= \left(\frac{n(T)}{1000}\right)\log_2\left(\frac{1000}{n(T)}\right) + \left(\frac{n(K)}{1000}\right)\log_2\left(\frac{1000}{n(K)}\right) + \left(\frac{n(M)}{1000}\right)\log_2\left(\frac{1000}{n(M)}\right) \\ H &= \left(\frac{n(T)}{1000}\right)\log_2\left(\frac{1000}{n(T)}\right) + \\ &\quad \left(\frac{2666.66 - 2.83n(T)}{1000}\right)\log_2\left(\frac{1000}{2666.66 - 2.83n(T)}\right) + \\ &\quad \left(\frac{1.83n(T) - 1666.66}{1000}\right)\log_2\left(\frac{1000}{1.83n(T) - 1666.66}\right) \end{aligned} \tag{8–15}$$

This graph is shown in Figure 8–2.

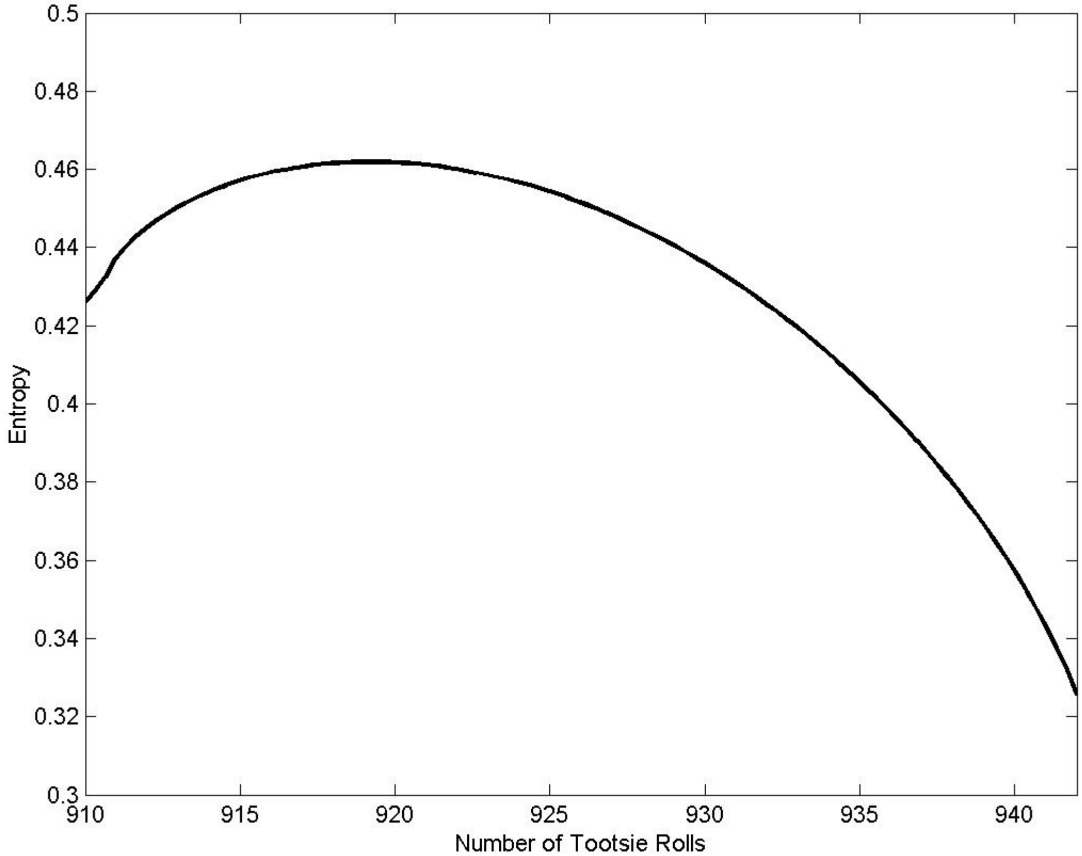

Figure 8–2: Entropy of the candy probability distribution

According to this graph, the maximum occurs at $n(T) = 919$, with an entropy of $H = 0.4619$. This gives $n(M) = 15$, and $n(K) = 66$.

Solution to Problem 2, part c.

The percentage mass of sugar is calculated via the following equation:

$$\begin{aligned} S &= 0.5 \times 2.5 \times n(T) + 0.64 \times 0.8 \times n(M) + 0.08 \times 1.4 \times n(K) \\ &= 1.25 \times 919 + 0.512 \times 15 + 0.112 \times 66 \\ &= 1148.75 + 7.68 + 7.392 \\ &= 1880.4 \text{ grams of sugar} \\ &= 78.3\% \text{ sugar} \end{aligned} \tag{8–16}$$

Solution to Problem 2, part d.

No, you cannot have 600 Tootsie Rolls. If you had 600 Tootsie Rolls (1500 g), would have at most 400 Hershey Kisses (560 g) for a total weight of 2160g, which does not satisfy the weight constraint of 2400g.