MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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6.050J/2.110J **Information and Entropy** **Spring 2004**

Problem Set 12 Solutions

Solution to Problem 1: Cheap Heat

Solution to Problem 1, part a.

You must run it in the reverse direction.

Solution to Problem 1, part b.

The outdoor temperature in Kelvin is 273.15 degrees, and the indoor temperature is 290.15 degrees Kelvin.

Solution to Problem 1, part c.

To find a relationship between T_1, T_2, H_c, and H_d we take the equation given and

$$T\,dS = \left(\sum_i p_i(E_i(H) - E)^2\right) \frac{1}{k_B T}\left(\frac{1}{T}dT - \frac{1}{H}dH\right) \tag{12–4}$$

but since $dS = 0$ the equation reduces to

$$\frac{1}{T}dT = \frac{1}{H}dH$$

integrating from c to d we have

$$\begin{aligned}
\int_{T_2}^{T_1} \frac{1}{T}dT &= \int_{H_c}^{H_d} \frac{1}{H}dH \\
\ln\left(\frac{T_1}{T_2}\right) &= \ln\left(\frac{H_d}{H_c}\right) \\
\frac{T_1}{T_2} &= \frac{H_d}{H_c} \qquad (12\text{–}5)
\end{aligned}$$

Solution to Problem 1, part d.

Thus finding H_d we have

$$\begin{aligned}
H_d &= H_c \frac{T_1}{T_2} \\
&= 1000 \times \frac{273.15}{290.15} \\
&= 941 \text{ A/m} \qquad (12\text{–}6) \\
& \qquad (12\text{–}7)
\end{aligned}$$

Solution to Problem 1, part e.

The heat extracted from outdoors is

$$Q = (S_2 - S_1)T_1 \tag{12–8}$$

The work done on the system is the heat pumped to the warm environment less the heat extracted from the cold environment

$$W = \frac{S_2 - S_1}{T_2 - T_1} \tag{12–9}$$

The coefficient of performance then is

$$\begin{aligned}\eta &= \frac{T_1}{T_2 - T_1}\\ &= \frac{273.15}{290.15 - 273.15}\\ &= 16.07\end{aligned} \tag{12–10}$$

Solution to Problem 1, part f.

Again, to find a relationship between T_1, T_2, H_a, and H_b we take the equation given

$$TdS = \left(\sum_i p_i(E_i(H) - E)^2\right)\frac{1}{k_B T}\left(\frac{1}{T}dT - \frac{1}{H}dH\right) \tag{12–11}$$

but since $dS = 0$ the equation reduces to

$$\frac{1}{T}dT = \frac{1}{H}dH$$

integrating from a to b we have

$$\begin{aligned}\int_{T_1}^{T_2}\frac{1}{T}dT &= \int_{H_a}^{H_b}\frac{1}{H}dH\\ \ln\left(\frac{T_2}{T_1}\right) &= \ln\left(\frac{H_b}{H_a}\right)\\ \frac{T_2}{T_1} &= \frac{H_b}{H_a}\end{aligned} \tag{12–12}$$

Solution to Problem 1, part g.

The magnetic field H_a is

$$\begin{aligned}H_a &= H_b\frac{T_1}{T_2}\\ &= 2000 \times \frac{273.15}{290.15}\\ &= 1883 \text{ A/m}\end{aligned} \tag{12–13}$$

$$\tag{12–14}$$

Solution to Problem 1, part h.

Since S is constant in this (adiabatic) leg, $dq = 0$.

To go further you have to calculate the probabilities, since you need them to find the energy E at each of the four corners. You already know the temperature and magnetic field at each corner, so it is straightforward to find α and then the probabilities using these equations from Chapter 12:

$$p_i = e^{-\alpha} e^{-E_i/k_B T} \tag{12–15}$$

$$\alpha = \ln\left(\sum_i e^{-E_i/k_B T}\right) \tag{12–16}$$

Solution to Problem 1, part i.

For corners a and b:

$$p_{up} = \frac{e^{m_{eff} H_a/k_B T_1}}{e^{m_{eff} H_a/k_B T_1} + e^{-m_{eff} H_a/k_B T_1}} \tag{12–17}$$

First calculate the exponential.

$$\begin{aligned} e^{m_{eff} H_a/k_B T_1} &= \exp\left(\frac{1.165 \times 10^{-24} \times 1883}{273.15 \times 1.38 \times 10^{-23}}\right) \\ &= \exp\left(\frac{2.1937 \times 10^{-21}}{3.77 \times 10^{-21}}\right) \\ &= \exp(0.5819) \\ &= 1.7894 \end{aligned} \tag{12–18}$$

Thus. . .

$$\begin{aligned} p_{up,a,b} &= \frac{1.7894}{1.7894 + \frac{1}{1.7894}} \\ &= 0.762 \end{aligned} \tag{12–19}$$

$$\begin{aligned} p_{down,a,b} &= 1 - p_{up} \\ &= 0.238 \end{aligned} \tag{12–20}$$

For corners c and d:

$$p_{up} = \frac{e^{m_{eff} H_d/k_B T_1}}{e^{m_{eff} H_d/k_B T_1} + e^{-m_{eff} H_d/k_B T_1}} \tag{12–21}$$

First calculate the exponential. . .

$$\begin{aligned} e^{m_{eff} H_a/k_B T_1} &= \exp\left(\frac{1.165 \times 10^{-24}}{\times} 941273.15 \times 1.38 \times 10^{-23}\right) \\ &= \exp\left(\frac{1.096 \times 10^{-21}}{3.77 \times 10^{-21}}\right) \\ &= \exp(0.2907) \\ &= 1.337 \end{aligned} \tag{12–22}$$

Thus...

$$\begin{aligned} p_{up,c,d} &= \frac{1.337}{1.337 + \frac{1}{1.337}} \\ &= 0.6413 \end{aligned} \quad (12\text{–}23)$$

$$\begin{aligned} p_{down,c,d} &= 1 - p_{up} \\ &= 0.3587 \end{aligned} \quad (12\text{–}24)$$

Solution to Problem 1, part j.

$$\begin{aligned} E_a &= \sum_i E_i p_i \\ &= -m_{eff} H_a p_{up,a} + m_{eff} H_a p_{down,a} \\ &= m_{eff} H_a (p_{down,a} - p_{up,a}) \\ &= -1.149 \times 10^{-21} Joules \end{aligned} \quad (12\text{–}25)$$

$$\begin{aligned} E_b &= \sum_i E_i p_i \\ &= -m_{eff} H_b p_{up,b} + m_{eff} H_b p_{down,b} \\ &= m_{eff} H_b (p_{down,b} - p_{up,b}) \\ &= -1.221 \times 10^{-21} Joules \end{aligned} \quad (12\text{–}26)$$

$$\begin{aligned} E_c &= \sum_i E_i p_i \\ &= -m_{eff} H_c p_{up,c} + m_{eff} H_c p_{down,c} \\ &= m_{eff} H_c (p_{down,c} - p_{up,c}) \\ &= -3.309 \times 10^{-21} Joules \end{aligned} \quad (12\text{–}27)$$

$$\begin{aligned} E_d &= \sum_i E_i p_i \\ &= -m_{eff} H_d p_{up,d} + m_{eff} H_d p_{down,d} \\ &= m_{eff} H_d (p_{down,d} - p_{up,d}) \\ &= -3.114 \times 10^{-21} Joules \end{aligned} \quad (12\text{–}28)$$

$$(12\text{–}29)$$

Solution to Problem 1, part k.

$$\begin{aligned} S_1 &= k_B \sum_i p_i \ln\left(\frac{1}{p_i}\right) \\ &= k_B \left(p_{up,a,b} \ln\left(\frac{1}{p_{up,a,b}}\right) + p_{down,a,b} \ln\left(\frac{1}{p_{down,a,b}}\right)\right) \\ &= 7.573 \times 10^{-24} \qquad (12\text{–}30) \\ S_2 &= k_B \sum_i p_i \ln\left(\frac{1}{p_i}\right) \\ &= k_B \left(p_{up,c,d} \ln\left(\frac{1}{p_{up,c,d}}\right) + p_{down,c,d} \ln\left(\frac{1}{p_{down,c,d}}\right)\right) \\ &= 9.001 \times 10^{-24} \qquad (12\text{–}31) \end{aligned}$$

therefore

$$S_2 - S_1 = 1.432 \times 10^{-24} \text{ Joules/Kelvin} \qquad (12\text{–}32)$$

Solution to Problem 1, part l.

$$\begin{aligned} dq_{ba} &= TdS \\ &= 0 \text{ Joules} \qquad (12\text{–}33) \\ dq_{ad} &= TdS \\ &= T_2(S_1 - S_2) \\ &= -4.142 \times 10^{-22} \text{ Joules} \qquad (12\text{–}34) \\ dq_{dc} &= TdS \\ &= 0 \text{ Joules} \qquad (12\text{–}35) \\ dq_{cb} &= TdS \\ &= T_1(S_2 - S_1) \\ &= 3.898 \times 10^{-22} \text{ Joules} \qquad (12\text{–}36) \\ & \qquad (12\text{–}37) \end{aligned}$$

Solution to Problem 1, part m.

$$\begin{aligned}
dw_{ba} &= dE_{ba} - dq_{ba} \\
&= E_b - E_a - 0 \\
&= -7.2 \times 10^{-23} \text{ Joules} \\
dw_{ad} &= dE_{ad} - dq_{ad} \\
&= E_a - E_d - dq_{ad} \\
&= -4.234 \times 10^{-22} \text{ Joules} \\
dw_{dc} &= dE_{dc} - dq_{dc} \\
&= E_d - E_c - 0 \\
&= -1.95 \times 10^{-23} \text{ Joules} \\
dw_{cb} &= dE_{cb} - dq_{cb} \\
&= E_c - E_b - dq_{cb} \\
&= 5.003 \times 10^{-22} \text{ Joules}
\end{aligned} \tag{12–38}$$

Solution to Problem 1, part n.

The work is the sum of the previous.

$$2.44 \times 10^{-23} \text{Joules} \tag{12–39}$$

Solution to Problem 1, part o.

$$16 \tag{12–40}$$

This is close to the coefficient.

Solution to Problem 0, part .

The number of Joules required to heat one gram of air one degree is

$$\frac{0.715}{1.277 \times 10^{-31}} = 5.59 \times 10^{30} \text{ cycles} \tag{12–41}$$

Solution to Problem 1, part p.

$$1.82 \times 10^{21} \text{ cycles} \tag{12–42}$$

Solution to Problem 2: Information is Cool

Solution to Problem 2, part a.

$$\frac{75 \text{ Calories/hour} \times 4.1868 \times 10^3 \text{ Joules/Calorie}}{3600 \text{ sec/hour}} = 87.225 \text{ Joules/sec} \tag{12–43}$$

People don't light up like lightbulbs because the energy they expend is distributed about the whole body, not concentrated on a microscopic filament.

Solution to Problem 2, part b.

A person needs to consume 75 Calories times 24 hours, or

$$75 \text{ Calories/hour} \times 24 \text{ hours/day} = 1800 \text{ Calories/day} \tag{12–44}$$

Solution to Problem 2, part c.

Mark burns an extra 825 Calories per week than a person who does no exercise. Paul consumes an extra 260 Calories every time he jobs, and thus burns only 45 Calories more than a person who does no exercise. Paul thus gains, in a year:

$$780 \text{ Calories/week} \times 52 \text{ weeks/year} = 40560 \text{ Calories/year} \tag{12–45}$$

John gains:

$$825 \text{ Calories/week} \times 52 \text{ weeks/year} = 42900 \text{ Calories/year} \tag{12–46}$$

Thus Paul gains:

$$\frac{40560 Calories \times 4.1868 \times 10^3 \text{ Joules/Calorie}}{33.1 \times 10^6 \text{ Joules/kg fat}} = 5.13 \text{ kg fat} \tag{12–47}$$

And John gains:

$$\frac{42900 Calories \times 4.1868 \times 10^3 \text{ Joules/Calorie}}{33.1 \times 10^6 \text{ Joules/kg fat}} = 5.42 \text{ kg fat} \tag{12–48}$$

Solution to Problem 2, part d.

The amount of heat the room is losing, in Watts, is:

$$\frac{5000 \times 10^3 \text{Joules/hour}}{3600 \text{ sec/hour}} = 1388 \text{ Watts} \tag{12–49}$$

If the temperature is to remain constant, the students and professor must produce the same amount of energy

$$200 + 120A + 80S = 1388 \text{ Watts} \tag{12–50}$$

where A = Awake and S = Sleeping. If the lecture has 25 students, then the sum of A and S equals 25 and so

$$\begin{aligned} 200 + 120A + 80(25 - A) &= 1388 \\ 2200 + 40A &= 1388 \\ A &= -20.3 \text{ students} \end{aligned} \tag{12–51}$$

We conclude that the temperature will always increase, since even with all the students asleep the amount of heat produced is greater than the heat dissipation.
which means that 20 students must be awake, 3 students asleep, and one student drifting in and out of consciousness, his head bobbing forward, waking himself up every so often, for an average of 66% of the time awake, 33% asleep.

Solution to Problem 2, part e.

The number of Calories consumed in raising 335 ml of water to body temperature (37 degrees Celsius) is

$$0.355 \text{ Liter} \times 1 \text{ Calories/Liter/degree C} \times 37\text{degrees C} = 13 \text{ Calories} \tag{12–52}$$

Only 7% of the Calories are consumed raising the rootbeer to body temperature. So Paul's argument is not correct.