Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
Department of Mechanical Engineering

6.050J/2.110J **Information and Entropy** **Spring 2004**

Problem Set 10 Solutions

Solution to Problem 1: Entropy Goes Up

Solution to Problem 1, part a.

The expectation of system energy E_s (the expected value of the energy) is calculated by the following formula.

$$\begin{aligned} E_s &= \sum_i p_{s,i} E_{s,i}(H) \\ &= 0\, m_d H - 1\, m_d H \\ &= -\, m_d H \end{aligned} \tag{11–1}$$

Solution to Problem 1, part b.

The expectation of the environment energy E_e is found in a similar manner.

$$\begin{aligned} E_e &= \sum_j p_{e,j} E_{e,j}(H) \\ &= 0.5\, m_d H - 0.5\, m_d H \\ &= 0 \end{aligned} \tag{11–2}$$

Solution to Problem 1, part c.

The environment entropy is calculated via the following formula.

$$\begin{aligned} S_e &= k_B \sum_j p_{e,j} \ln\left(\frac{1}{p_{e,j}}\right) \\ &= 0.693 k_B \end{aligned} \tag{11–3}$$

Solution to Problem 1, part d.

The system entropy S_s is calculated in a similar fashion.

$$\begin{aligned} S_s &= k_B \sum_i p_{s,i} \ln\left(\frac{1}{p_{s,i}}\right) \\ &= k_B \left(0.5 \ln\left(\frac{1}{0.5}\right) + 0.5 \ln\left(\frac{1}{0.5}\right)\right) \\ &= 0 \end{aligned} \tag{11–4}$$

Solution to Problem 1, part e.

No energy leaves the system and environment combined (by definition) so the expectation of the total energy is just the sum of the expectations of the energy of the system and environment.

$$\begin{aligned} E_t &= E_s + E_e \\ &= -m_d H \end{aligned} \tag{11–5}$$

Solution to Problem 1, part f.

To find β_t we first use Equation 10.20. Note that $E_{i,j}$=-2, 0,0, or 2 times $m_d H$.

$$\begin{aligned} 0 &= \sum_{i,j} (E_{i,j} - E_t)\, e^{-\beta_t E_{i,j}} \\ &= \sum_{i,j} E_{i,j} e^{-\beta_t E_{i,j}} - E_t \sum_{i,j} e^{-\beta_t E_{i,j}} \\ &= m_d H \left(e^{m_d H \beta_t} + 2 + e^{-2 m_d H \beta_t} \right) \end{aligned} \tag{11–6}$$

Therefore $\beta_t = \ln(3)/2 m_d H$.

Solution to Problem 1, part g.

The probabilities are defined as

$$p_{i,j} = \frac{e^{-\beta_t E_{i,j}}}{\sum_{i,j} e^{-\beta_t E_{i,j}}} \tag{11–7}$$

Thus

So

$$p_{0,0} = 9/16 \tag{11–8}$$
$$p_{0,1} = 3/16 \tag{11–9}$$
$$p_{1,0} = 3/16 \tag{11–10}$$
$$p_{1,1} = 1/16 \tag{11–11}$$
(11–12)

Solution to Problem 1, part h.

The total entropy is

$$\begin{aligned} S_t &= k_B \sum_i p_i \ln\left(\frac{1}{p_i}\right) \\ &= 1.125 k_B \end{aligned} \tag{11–13}$$

which is higher than the original entropy, $0.693 k_B$

Solution to Problem 1, part i.

First let us infer from the four probabilities for the total configuration $p_{t,i,j}$ the probabilities for the two system states $p_{s,i}$.

The energy is

$$\begin{aligned} E_s &= \sum_i p_{s,i} E_{s,i}(H) \\ &= -0.5\, m_d H \end{aligned} \tag{11–14}$$

Thus we see that exactly half the energy is in the system.

Solution to Problem 1, part j.

The system started out with $-m_d H$ Joules in it, and ended up with $-m_d H/2$ Joules in it. Thus $m_d H/2$ Joules flowed from the environment to the system.