Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
Department of Mechanical Engineering

6.050J/2.110J **Information and Entropy** **Spring 2004**

Final Exam Solutions

Solution to Problem 1: Information and Jeopardy (9%)

Solution to Problem 5, part a.

This physicist of the 19th century has a unit of temperature named after him. Who was ________Kelvin________?

Solution to Problem 5, part b.

This mathematician, whose name sounds suspiciously edible, invented a measure of how far two codewords are from each other. Who was ________Hamming________?

Solution to Problem 5, part d.

He was one of three after whom are named a compression technique used in the popular GIF image format. Who was ____Lempel/Ziv/Welsh____?

Solution to Problem 1, part d.

This Yale mathematician received the first doctorate in engineering awarded in America, and has a famous inequality that bears his name. Who was ________Gibbs________?

Solution to Problem 1, part e.

After reportedly secluding himself in his mountain cabin, with pearls in his ears to muffle the sound, and his girlfriend in his bed to inspire him, this famous physicist came up with an equation which bears his name that is used to calculate the wavefunction of a quantum system. Who was ________Schrödinger________?

Solution to Problem 1, part f.

The constant named after this physicist, who committed suicide in 1906 purportedly because his scientific ideas were not accepted by his peers, has a value of approximately 1.38×10^{-23} Joules per Kelvin. Who was________Boltzmann________?

Solution to Problem 1, part g.

This military engineer showed, through the quantity that bears his name, that the efficiency of a heat engine can never be 1. Who was ________Carnot________?

Solution to Problem 1, part h.

Known most famously for the form of algebra named after him, this mathematician was a childhood prodigy in Latin, publishing at the age of 12. Who was ________Boole________?

Solution to Problem 1, part i.

While at MIT as a graduate student in the 1940's this man invented a famous inequality which bears his name. Who was ________Kraft________?

Solution to Problem 2: MIT Customer Complaint Department (15%)

Solution to Problem 2, part a.

Design a Huffman code for the complaints above. To design a Huffman code, we have to recursively group the two subgroups of complaints with smallest probability.

Complaint	Code
Not enough homework	1
Campus dining options too diverse	01
Tuition too low	001
Administration too attentive	0001
Classes too easy	0000

Solution to Problem 2, part b.

What is the average number of bits to send one complaint? Calculating the expectation of the number of bits: $0.5 \times 1 + 0.3 \times 2 \times 0.1 \times 3 + 0.05 \times 4 + 0.05 \times 4 = 0.5 + 0.6 + 0.3 + 0.4 = 1.8$

Average # of bits/complaint: ________1.8________

Solution to Problem 3: The Traveling SailMan (20%)

Solution to Problem 3, part a.

With this knowledge, what values of S, M, and L are possible?

We know that S and M can reach zero, since they are less than the average.

S_{min} __0__ S_{max} __1/3__ M_{min} __0__ M_{max} __1/2__ L_{min} __1/2__ L_{max} __2/3__

Solution to Problem 3, part b.

You decided to use the Principle of Maximum Entropy to estimate S, M, L, and the resulting uncertainty U about the number deployed. Express the entropy as a function of a single probability (any one of the three, S, M, or L).

$$\begin{aligned}
\text{Entropy}(S) &= S\log_2\left(\frac{1}{S}\right) + \frac{1-3S}{2}\log_2\left(\frac{2}{1-3S}\right) + \frac{S+1}{2}\log_2\left(\frac{2}{S+1}\right) \\
\text{Entropy}(M) &= \frac{1-2M}{3}\log_2\left(\frac{3}{1-2M}\right) + M\log_2\left(\frac{1}{M}\right) + \frac{2-M}{3}\log_2\left(\frac{3}{2-M}\right) \\
\text{Entropy}(L) &= (2L-1)\log_2\left(\frac{1}{2L-1}\right) + (2-3L)\log_2\left(\frac{1}{2-3L}\right) + L\log_2\left(\frac{1}{L}\right)
\end{aligned}$$

Solution to Problem 3, part c.

What probabilities S, M, and L did he want readers to infer, and what would be their resulting uncertainty in bits about the number of reconnaissance robots deployed in any one marina?

$S =$ 1/3 $M =$ 0 $L =$ 2/3 Uncertainty = 0.92

Solution to Problem 3, part d.

Compare this uncertainty to the value of U which you started to find earlier in this problem. Is this uncertainty

X Less than U? ____ Equal to U? ____ Greater than U?

Solution to Problem 4: Variations on a Theme by Carnot (25%)

Solution to Problem 4, part a.

Does a heat engine traverse the rectangle in the above figure clockwise or counterclockwise?

CW or CCW? CCW

Solution to Problem 4, part b.

Give expressions for the cost, benefit and efficiency of the heat engine in terms of the quantities S_1, S_2, T_c, and T_h in the figure above.

$$\text{Cost} = Q_a = T_h(S_2 - S_1)$$

$$\text{Benefit} = Q_a - Q_b = (T_h - T_c)(S_2 - S_1)$$

$$\text{Efficiency} = \frac{\text{Benefit}}{\text{Cost}} = \frac{Q_a - Q_b}{Q_a} = \frac{T_h - T_c}{T_h}$$

Solution to Problem 4, part c.

Does a refrigerator traverse the rectangle in the above figure clockwise or counterclockwise?

CW or CCW? CW

Solution to Problem 4, part d.

Give expressions for the cost, benefit and coefficient of performance of the refrigerator in terms of the quantities S_1, S_2, T_c, and T_h in the figure above.

$$\text{Cost} = Q_d - Q_c = (T_h - T_c)(S_2 - S_1)$$

$$\text{Benefit} = Q_c = T_h(S_2 - S_1)$$

$$\text{Coefficient of Performance} = \frac{\text{Benefit}}{\text{Cost}} = \frac{Q_c}{Q_d - Q_c} = \frac{T_h}{T_h - T_c}$$

Solution to Problem 4, part e.

Does a heat pump traverse the rectangle in the figure above clockwise or counterclockwise?

CW or CCW? CW

Solution to Problem 4, part f.

Give expressions for the cost, benefit and coefficient of performance of the heat pump in terms of the quantities S_1, S_2, T_c, and T_h in the figure above.

$$\text{Cost} = Q_f - Q_e = (T_h - T_c)(S_2 - S_1)$$

$$\text{Benefit} = Q_f = T_h(S_2 - S_1)$$

$$\text{Coefficient of Performance} = \frac{\text{Benefit}}{\text{Cost}} = \frac{Q_f}{Q_f - Q_e} = \frac{T_h}{T_h - T_c}$$

Solution to Problem 5: The Telephone Game (30%)

Solution to Problem 5, part a.

$p(x_0 = 0) =$ 0.5 $p(x_0 = 1) =$ 0.5 $I_0 =$ 1 bits

$p(x_1 = 0) =$ 0.5 $p(x_1 = 1) =$ 0.5 $I_1 =$ 1 bits

$N =$ 0.72 $L =$ 0.72 $M =$ 0.28

Solution to Problem 5, part b.

$p(x_0 = 0) =$ 1 $p(x_0 = 1) =$ 0 $I_0 =$ 0 bits

$p(x_1 = 0) =$ 0.8 $p(x_1 = 1) =$ 0.2 $I_1 =$ 0.72 bits

$N =$ 0.72 $L =$ 0 $M =$ 0

Solution to Problem 5, part c.

Is it true that $p(x_k = 0) > 0.5$ for every channel output x_k? In other words, is every value passed along more likely to be 0 than 1? Write a paragraph defending your conclusion.

Lets begin by noting that $p(x_0 = 1) = 1$, which is greater than $1/2$. We need to show that $p(x_k = 0) > 0.5$ implies that $p(x_{k+1} = 0) > 0.5$ for all $k \geq 0$, which will guarantee by induction that $p(x_k = 0) > 0.5$ for all $k \geq 0$.

Let $p(x_k = 0)$ be written as p_k^0 for brevity. From basic probability,

$$\begin{aligned} p_{k+1}^0 &= p(x_{k+1} = 0 | x_k = 0) p_k^0 + p(x_{k+1} = 0 | x_k = 1)(1 - p_k^0) \\ &= (0.8) p_k^0 + (0.2)(1 - p_k^0) \\ &= 0.2 + 0.6 p_k^0 \end{aligned} \tag{F–3}$$

We can see algebraically that if $p_k^0 > 0.5$ then $p_{k+1}^0 > 0.5$. It is also visually clear from the plot of the function we derived above.

Solution to Problem 5, part d.

You conclude (correctly) that $p(x_k = 0)$ decreases as the message moves along the chain, i.e.,

$$p(x_0 = 0) > p(x_1 = 0) > p(x_2 = 0) > \ldots > p(x_k = 0) > p(x_{k+1} = 0) > \ldots \tag{F–4}$$

Let I_k be your uncertainty about x_k in bits. Does the uncertainty about x_k increase as you move through successive channels? In other words, is the following sequence of inequalities true?

$$I_0 < I_1 < I_2 < \ldots < I_k < I_{k+1} < \ldots \tag{F–5}$$

Write a paragraph defending your answer. We have seen the plot of the entropy of a two-state system in the course notes. Since the slope of this plot is negative for $p_k^0 > 0.5$, and since $p_k^0 > p_{k+1}^0 > 0.5$ for all k, i.e., the sequence of probabilities that $x_k = 0$ is decreasing but greater than a half, it follows that $I_k < I_{k+1}$, i.e., the sequence of uncertainties is increasing.

We can also verify analytically that the slope is negative for $p_k^0 > 0.5$:

$$\begin{aligned}
I_k &= -[p_k^0 \log_2(p_k^0) + (1 - p_k^0)\log_2(1 - p_k^0)] \\
&= -(1.443)[p_k^0 \ln(p_k^0) + (1 - p_k^0)\ln(1 - p_k^0)] \\
\frac{dI_k}{dp_k^0} &= -(1.443)[1 + \ln(p_k^0) - 1 - \ln(1 - p_k^0)] \\
&= -(1.443)\ln\left(\frac{p_k^0}{1 - p_k^0}\right) \\
&< 0, \text{ for } p_k^0 > 0.5
\end{aligned}$$