Name (1%): ____

Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science
Department of Mechanical Engineering

6.050J/2.110J **Information and Entropy** **Spring 2004**

Issued: May 21, 2004, 1:30 PM **Final Exam** **Due: May 21, 2004, 4:30 PM**

Note: Please write your name at the top of each page in the space provided. The last page may be removed and used as a reference table for the calculation of logarithms.

Problem 1: Information and Jeopardy (9%)

From the list of names from Table 1 below, fill in the blank with the name that best fits each of the following questions. Make sure your answers to each part are different.

a. This physicist of the 19th century has a unit of temperature named after him. Who was ____________________?

b. This mathematician, whose name sounds suspiciously edible, invented a measure of how far two codewords are from each other. Who was ____________________?

c. He was one of three after whom are named a compression technique used in the popular GIF image format. Who was ____________________?

d. This Yale mathematician received the first doctorate in engineering awarded in America, and has a famous inequality that bears his name. Who was ____________________?

e. After reportedly secluding himself in his mountain cabin, with pearls in his ears to muffle the sound, and his girlfriend in his bed to inspire him, this famous physicist came up with an equation which bears his name that is used to calculate the wavefunction of a quantum system. Who was ____________________?

f. The constant named after this physicist, who committed suicide in 1906 purportedly because his scientific ideas were not accepted by his peers, has a value of approximately 1.38×10^{-23} Joules per Kelvin. Who was____________________?

g. This military engineer showed, through the quantity that bears his name, that the efficiency of a heat engine can never be 1. Who was ____________________?

h. Known most famously for the form of algebra named after him, this mathematician was a childhood prodigy in Latin, publishing at the age of 12. Who was ____________________?

i. While at MIT as a graduate student in the 1940's this man invented a famous inequality which bears his name. Who was ____________________?

Avogadro	Bayes	Boltzmann	Boole	Carnot	Gibbs	Hamming
Huffman	Jaynes	Joule	Kelvin	Kraft	Lempel	Maxwell
Morse	Reed	Schrödinger	Shannon	Solomon	Welsh	Ziv

Table 1: List of Names

Problem 2: MIT Customer Complaint Department (15%)

You have recently been elected President of the UA, and it is your job to transmit student complaints to Chuck Vest so they (hopefully) can be addressed before he steps down. According to the UA's research, all student complaints fall into one of six categories, with percentages shown:

%	Complaint
50%	Not enough homework
30%	Campus dining options too diverse
10%	Tuition too low
5%	Administration too attentive
5%	Classes too easy

Unfortunately, Vest doesn't have much time, so he instructs you to only send very short messages to him. Because you've taken 6.050 you know about coding schemes, so you decide to encode the complaints above in a Huffman code.

a. Design a Huffman code for the complaints above.

Complaint	Code
Not enough homework	________
Campus dining options too diverse	________
Tuition too low	________
Administration too attentive	________
Classes too easy	________

b. What is the average number of bits to send one complaint?

Average # of bits/complaint: ____________________

Problem 3: The Traveling SailMan (20%)

Nothing annoys owners of expensive sailboats more than dirty sails. The SailMan robot solves this problem. Every night it swims from boat to boat, climbs aboard, and cleans the sails. This robot is helped by reconnaissance robots that report the location of boats with dirty sails to a central computer, which then plans the night's activities by solving the "Traveling SailMan" problem. Because of software limitations, reconnaissance robots can only be deployed in groups of 2 (Small), 4 (Medium), or 8 (Large).

Visiting a marina one evening, you noticed the SailMan robot and naturally wondered how many reconnaissance robots were deployed there. You expressed your knowledge in terms of probabilities (call them S, M, and L) that there were 2, 4, or 8 reconnaissance robots. You remembered from the SailMan Annual Report that the average number of reconnaissance robots deployed in a marina was 6.

a. With this knowledge, what values of S, M, and L are possible?

S_{min}________ S_{max}________ M_{min}________ M_{max}________ L_{min}________ L_{max}________

b. You decided to use the Principle of Maximum Entropy to estimate S, M, L, and the resulting uncertainty U about the number deployed. Express the entropy as a function of a single probability (any one of the three, S, M, or L).

Entropy = ____

Before you had a chance to evaluate U, you happened to run into the the Fiscal Officer of SailMan, Inc., who said that when he wrote the Annual Report he wanted readers to think the high-priced configuration of 8 reconnaissance robots was selling better than it really was. He reported the average number of reconnaissance robots (6) correctly but then mentioned the largest value of L that was consistent with this average, without actually saying it was correct (linguists would call this an "intended inference").

c. What probabilities S, M, and L did he want readers to infer, and what would be their resulting uncertainty in bits about the number of reconnaissance robots deployed in any one marina?

$S =$ __________ $M =$ __________ $L =$ __________ Uncertainty = __________

d. Compare this uncertainty to the value of U which you started to find earlier in this problem. Is this uncertainty

_______ Less than U? _______ Equal to U? _______ Greater than U?

Problem 4: Variations on a Theme by Carnot (25%)

The ideal (and most efficient) version of a magnetic heat engine operates reversibly in a cycle that can be represented as a rectangle in the $T-S$ plane. The same physical device can also be operated reversibly as a refrigerator or as a heat pump to accomplish different goals. This problem asks you to quantify the effectiveness of these three systems in terms of the variables in the diagram below. The entropies and temperatures in the plot and the energies given in the description of each problem are all positive.

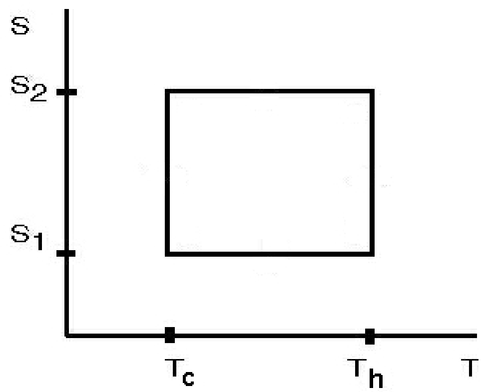

Figure 1: $T-S$ Cycle for a reversible heat engine

Heat Engine: The purpose of a heat engine is to convert heat to work. In each cycle a heat engine reversibly extracts Q_a Joules of heat from a hot reservoir at temperature T_h, dumps a portion Q_b Joules of that heat into a cooler environment at temperature T_c and converts the remaining energy into $(Q_a - Q_b)$ Joules of useful work. The cost per cycle is the energy Q_a taken from the hot reservoir and the benefit per cycle is the work $(Q_a - Q_b)$ performed.

a. Does a heat engine traverse the rectangle in the above figure clockwise or counterclockwise?

CW or CCW? ____________________

b. Give expressions for the cost, benefit and efficiency of the heat engine in terms of the quantities S_1, S_2, T_c, and T_h in the figure above.

$\text{Cost} = Q_a =$ ____________________

$\text{Benefit} = Q_a - Q_b =$ ____________________

$\text{Efficiency} = \frac{\text{Benefit}}{\text{Cost}} = \frac{Q_a - Q_b}{Q_a} =$ ____________________

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Refrigerator: The purpose of a refrigerator is to cool something below the temperature of its environment. Though the temperature of the object to be cooled will drop over time, the ideal model represents the cooled object as a constant-temperature object at a temperature T_c. In each cycle a refrigerator reversibly extracts Q_c Joules of heat from a cold object, ejects a larger quantity Q_d Joules of heat into a warmer environment at temperature T_h, and requires $(Q_d - Q_c)$ Joules of work to traverse the cycle. The cost per cycle is the $(Q_d - Q_c)$ Joules of work that must be performed and the benefit per cycle is the Q_c Joules of heat taken from the cold object.

c. Does a refrigerator traverse the rectangle in the above figure clockwise or counterclockwise?

CW or CCW? ____________________

d. Give expressions for the cost, benefit and coefficient of performance of the refrigerator in terms of the quantities S_1, S_2, T_c, and T_h in the figure above.

Cost $= Q_d - Q_c =$ ____________________

Benefit $= Q_c =$ ____________________

Coefficient of Performance $= \frac{\text{Benefit}}{\text{Cost}} = \frac{Q_c}{Q_d - Q_c} =$ ____________________

Heat Pump: The purpose of a heat pump is to warm something (e.g., your house) efficiently by making use of heat that is available at a lower temperature (e.g., an underground well on your property.) Though the temperature of your house will rise over time, the ideal model represents it as a constant-temperature reservoir at a temperature T_h. In each cycle a heat pump reversibly extracts Q_e Joules of heat from a cold reservoir at temperature T_c, ejects a larger quantity Q_f Joules of heat into a warmer environment and requires $(Q_f - Q_e)$ Joules of work to traverse the cycle. The cost per cycle is the $(Q_f - Q_e)$ Joules of work that must be performed and the benefit per cycle is the Q_f Joules of heat added to the hot reservoir.

e. Does a heat pump traverse the rectangle in the figure above clockwise or counterclockwise?

CW or CCW? ____________________

f. Give expressions for the cost, benefit and coefficient of performance of the heat pump in terms of the quantities S_1, S_2, T_c, and T_h in the figure above.

Cost $= Q_f - Q_e =$ ____________________

Benefit $= Q_f =$ ____________________

Coefficient of Performance $= \frac{\text{Benefit}}{\text{Cost}} = \frac{Q_f}{Q_f - Q_e} =$ ____________________

Problem 5: The Telephone Game (30%)

The "Telephone Game" illustrates how correct information gets converted into false rumors. In the game one person (Alice) sends a message to another (Bob) through a chain of humans, each of whom potentially corrupts the message. Thus Alice tells person #1, who then tells person #2, and so on until the last person in the chain tells Bob. Then Bob and Alice announce their versions of the message, normally accompanied by amazement at how different they are.

Consider the case where a single bit is being passed and each person in the chain has a 20% probability of passing on a bit different from the one she received. Thus we model each person in the chain as a symmetric binary channel as shown in Figure 1.

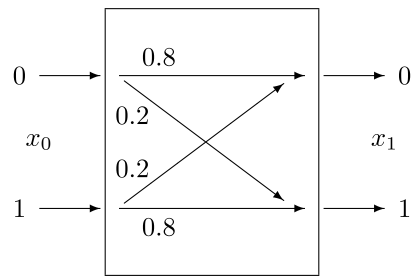

Figure 1: Simple model of a person

The game is being demonstrated for you at a party one day. Alice and Bob take their positions at opposite ends of the chain, and Alice whispers the value of x_0 to person #1. You know that person #1, like the other members of the chain, has a 20% probability of changing the bit she hears. Parts *a.* and *b.* concern the model for this person, and parts *c.* and *d.* concern the behavior of the chain.

a. At first, you do not know what Alice has told person #1. Naturally, you express your state of knowledge in terms of the two probabilities $p(x_0 = 0)$ and $p(x_0 = 1)$. To avoid any unintended bias you use the Principle of Maximum Entropy to conclude that each of these probabilities is equal to 0.5. Then you calculate your uncertainty I_0 about the value of x_0, the output probabilities $p(x_1 = 0)$ and $p(x_1 = 1)$, and your uncertainty I_1 about the value of x_1. Then you calculate the channel noise N, channel loss L, and mutual information M, all in bits:

$p(x_0 = 0) =$ ____0.5____ $p(x_0 = 1) =$ ____0.5____ $I_0 =$ ______________ bits

$p(x_1 = 0) =$ __________ $p(x_1 = 1) =$ __________ $I_1 =$ ______________ bits

$N =$ ______________ $L =$ ______________ $M =$ ______________

b. Then you happen to see the sheet of paper Alice is holding, and you discover that $x_0 = 0$. You decide to recalculate all the quantities so they are consistent with your new state of knowledge:

$p(x_0 = 0) =$ ____1____ $p(x_0 = 1) =$ ____0____ $I_0 =$ ______________ bits

$p(x_1 = 0) =$ __________ $p(x_1 = 1) =$ __________ $I_1 =$ ______________ bits

$N =$ ______________ $L =$ ______________ $M =$ ______________

Next, consider the behavior of a cascade of independent identical channels representing the individuals passing the message, as illustrated in Figure 2. Let x_k represent the output of the k-th channel.

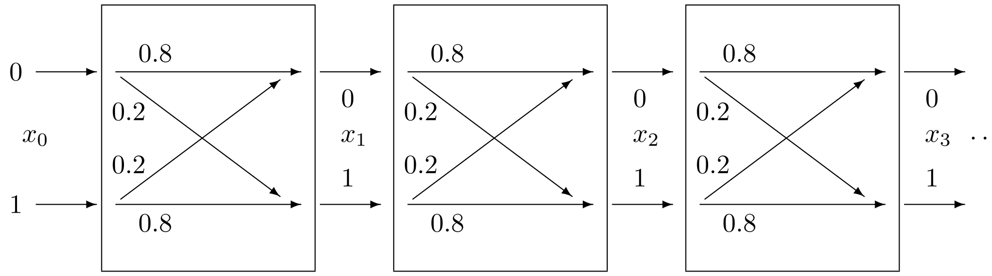

Figure 2: Simple model of the telephone game

c. Knowing that $x_0 = 0$, and having just calculated probabilities for x_1, you wonder how much you know about the other values, x_2, x_3, x_4, $\ldots x_k \ldots$. Is it true that $p(x_k = 0) > 0.5$ for every channel output x_k? In other words, is every value passed along more likely to be 0 than 1? Write a paragraph defending your conclusion. If possible, make this an outline of a proof. You may find some version of the principle of induction helpful. (Pictures and equations are allowed in the paragraph, if needed.)

d. You conclude (correctly) that $p(x_k = 0)$ decreases as the message moves along the chain, i.e.,

$$p(x_0 = 0) > p(x_1 = 0) > p(x_2 = 0) > \ldots > p(x_k = 0) > p(x_{k+1} = 0) > \ldots \tag{1}$$

Let I_k be your uncertainty about x_k in bits. Does the uncertainty about x_k increase as you move through successive channels? In other words, is the following sequence of inequalities true?

$$I_0 < I_1 < I_2 < \ldots < I_k < I_{k+1} < \ldots \tag{2}$$

Write a paragraph defending your answer.

Logarithm and Entropy Table

This page is provided so that you may rip it off the exam to use as a separate reference table. In Table 1, the entropy $S = p \log_2(1/p) + (1-p)\log_2(1/(1-p))$.

p	1/8	1/5	1/4	3/10	1/3	3/8	2/5	1/2	3/5	5/8	2/3	7/10	3/4	4/5	7/8
$\log_2(1/p)$	3.00	2.32	2.00	1.74	1.58	1.42	1.32	1.00	0.74	0.68	0.58	0.51	0.42	0.32	0.18
S	0.54	0.72	0.81	0.88	0.92	0.95	0.97	1.00	0.97	0.95	0.92	0.88	0.81	0.72	0.54

Table 1: Table of logarithms in base 2 and entropy in bits