

## Problem Set 8 Solutions

### Solution to **Problem 1: Berger's Burgers Strikes Back**

#### Solution to Problem 1, part a.

The corresponding equations in your case are:

$$0 = B + C + F + T - 1 \quad (8-18)$$

$$0 = 1000B + 600C + 400F + 200T - 380 \quad (8-19)$$

#### Solution to Problem 1, part b.

The corresponding equations in your case are:

$$B = 2^{-\alpha}2^{-1000\beta} \quad (8-20)$$

$$C = 2^{-\alpha}2^{-600\beta} \quad (8-21)$$

$$F = 2^{-\alpha}2^{-400\beta} \quad (8-22)$$

$$T = 2^{-\alpha}2^{-200\beta} \quad (8-23)$$

Notice that at this point we have six equations for six unknowns ( $\alpha$ ,  $\beta$ ,  $B$ ,  $C$ ,  $F$ , and  $T$ ).

#### Solution to Problem 1, part c.

Start with the first constraint, Equation 8-18, and substitute in Equations 8-20 to 8-23:

$$\begin{aligned} 1 &= B + C + F + T \\ &= 2^{-\alpha}2^{-1000\beta} + 2^{-\alpha}2^{-600\beta} + 2^{-\alpha}2^{-400\beta} + 2^{-\alpha}2^{-200\beta} \\ &= 2^{-\alpha}(2^{-1000\beta} + 2^{-600\beta} + 2^{-400\beta} + 2^{-200\beta}) \end{aligned} \quad (8-24)$$

multiply both sides by  $2^\alpha$ :

$$2^\alpha = 2^{-1000\beta} + 2^{-600\beta} + 2^{-400\beta} + 2^{-200\beta} \quad (8-25)$$

and take the logarithm of both sides

$$\alpha = \log_2(2^{-1000\beta} + 2^{-600\beta} + 2^{-400\beta} + 2^{-200\beta}) \quad (8-26)$$

**Solution to Problem 1, part d.**

Start with the second constraint, Equation 8-19, and substitute in Equations 8-20 to 8-23:

$$\begin{aligned} 380 &= 1000B + 600C + 400F + 200T \\ &= 1000 \times 2^{-\alpha} 2^{-1000\beta} + 600 \times 2^{-\alpha} 2^{-600\beta} + 400 \times 2^{-\alpha} 2^{-400\beta} + 200 \times 2^{-\alpha} 2^{-200\beta} \end{aligned} \quad (8-27)$$

multiply both sides by  $2^\alpha$  and substitute Equation 8-25:

$$\begin{aligned} 380 \times 2^\alpha &= 380(2^{-1000\beta} + 2^{-600\beta} + 2^{-400\beta} + 2^{-200\beta}) \\ &= 1000 \times 2^{-1000\beta} + 600 \times 2^{-600\beta} + 400 \times 2^{-400\beta} + 200 \times 2^{-200\beta} \end{aligned} \quad (8-28)$$

subtract the lefthand side from the righthand side:

$$0 = 620 \times 2^{-1000\beta} + 220 \times 2^{-600\beta} + 20 \times 2^{-400\beta} - 180 \times 2^{-200\beta} \quad (8-29)$$

and multiply both sides by  $2^{380\beta}$  and distribute

$$\begin{aligned} 0 &= 2^{380\beta} (620 \times 2^{-1000\beta} + 220 \times 2^{-600\beta} + 20 \times 2^{-400\beta} - 180 \times 2^{-200\beta}) \\ &= 620 \times 2^{-620\beta} + 220 \times 2^{-220\beta} + 20 \times 2^{-20\beta} - 180 \times 2^{180\beta} \end{aligned} \quad (8-30)$$

so  $\beta$  is the value for which  $f(\beta) = 0$  where:

$$f(\beta) = 620 \times 2^{-620\beta} + 220 \times 2^{-220\beta} + 20 \times 2^{-20\beta} - 180 \times 2^{180\beta} \quad (8-31)$$

**Solution to Problem 1, part e.**

Starting with Equation 8-31 and substituting  $\beta = 0$ :

$$\begin{aligned} f(\beta) &= 620 \times 2^{-620\beta} + 220 \times 2^{-220\beta} + 20 \times 2^{-20\beta} - 180 \times 2^{180\beta} \\ &= 620 \times 2^{-620 \times 0} + 220 \times 2^{-220 \times 0} + 20 \times 2^{-20 \times 0} - 180 \times 2^{180 \times 0} \\ &= 620 + 220 + 20 - 180 \\ &= 680 \end{aligned} \quad (8-32)$$

Thus in your case,  $f(0) = 680$  Calories, which is positive.

**Solution to Problem 1, part f.**

$$\begin{aligned} f(0.01) &= 620x2^{-6.2} + 220x2^{-2.2} + 20x2^{-0.2} - 180x2^{1.8} \\ &= 8.43 + 47.88 + 17.41 - 626.80 \\ &= -553.08 \text{ Calories} \end{aligned} \quad (8-33)$$

Which is negative. Thus we have to decrease  $\beta$ . To find the zero crossing of  $f(b)$  you can use the following MATLAB code:

```
gB = 1000; gC = 600; gF = 400; gT = 200; G = 380;
b = sym('b');
f = (gB - G)*2^[b*(G-gB)] + (gC - G)*2^[b*(G-gC)] + ...
    (gF - G)*2^[b*(G-gF)] + (gT - G)*2^[b*(G-gT)]; b = fzero(inline(char(f)),0)
```

$$b = 0.0036$$

Note:  $\beta$  is expressed in bits per Calorie.

### Solution to Problem 1, part g.

Using Equation 8-26:

$$\alpha = \log_2(2^{-1000\beta} + 2^{-600\beta} + 2^{-400\beta} + 2^{-200\beta})$$

and substituting  $\beta = 0.0036$ :

$$\begin{aligned} &= \log_2(2^{-1000 \times 0.0036} + 2^{-600 \times 0.0036} + 2^{-400 \times 0.0036} + 2^{-200 \times 0.0036}) \\ &= \log_2(2^{-3.6} + 2^{-2.16} + 2^{-1.44} + 2^{-0.72}) \\ &= \log_2(0.0824 + 0.2238 + 0.3686 + 0.6071) \\ &= 0.3583 \end{aligned}$$

(8-34)

### Solution to Problem 1, part h.

Starting with Equations 8-20 to 8-23, and substituting in  $\alpha$  and  $\beta$ :

$$\begin{aligned} B &= 2^{-\alpha} 2^{-1000\beta} \\ &= 2^{-0.3583} 2^{-1000 \times 0.0036} \\ &= 0.7801 \times 0.0824 \\ &= 0.0643 \end{aligned} \tag{8-35}$$

$$\begin{aligned} C &= 2^{-\alpha} 2^{-600\beta} \\ &= 2^{-0.3583} 2^{-600 \times 0.0036} \\ &= 0.7801 \times 0.2238 \\ &= 0.1746 \end{aligned} \tag{8-36}$$

$$\begin{aligned} F &= 2^{-\alpha} 2^{-400\beta} \\ &= 2^{-0.3583} 2^{-400 \times 0.0036} \\ &= 0.7801 \times 0.3686 \\ &= 0.2875 \end{aligned} \tag{8-37}$$

$$\begin{aligned} T &= 2^{-\alpha} 2^{-200\beta} \\ &= 2^{-0.3583} 2^{-200 \times 0.0036} \\ &= 0.7801 \times 0.6071 \\ &= 0.4736 \end{aligned} \tag{8-38}$$

(8-39)

### Solution to Problem 1, part i.

$$\begin{aligned} B + C + F + T &= 0.0643 + 0.1746 + 0.2875 + 0.4736 \\ &= 1 \end{aligned} \tag{8-40}$$

**Solution to Problem 1, part j.**

$$\begin{aligned}
1000B + 600C + 400F + 200T &= 1000 \times 0.0643 + 600 \times 0.1746 + 400 \times 0.2875 + 200 \times 0.4736 \\
&= 64.3 + 104.76 + 115 + 94.72 \\
&= 378.78
\end{aligned} \tag{8-41}$$

**Solution to Problem 1, part k.**

The entropy  $S$  is calculated from the probabilities:

$$\begin{aligned}
S &= B \log_2 \left( \frac{1}{B} \right) + C \log_2 \left( \frac{1}{C} \right) + F \log_2 \left( \frac{1}{F} \right) + T \log_2 \left( \frac{1}{T} \right) \\
&= 0.0643 \log_2 \left( \frac{1}{0.0643} \right) + 0.1746 \log_2 \left( \frac{1}{0.1746} \right) + 0.2875 \log_2 \left( \frac{1}{0.2875} \right) + 0.4736 \log_2 \left( \frac{1}{0.4736} \right) \\
&= 0.0643 \log_2 (15.5521) + 0.1746 \log_2 (5.7274) + 0.2875 \log_2 (3.4783) + 0.4736 \log_2 (2.1115) \\
&= 0.0643 \times 3.9590 + 0.1746 \times 2.5179 + 0.2875 \times 1.7984 + 0.4736 \times 1.0783 \\
&= 0.2546 + 0.4396 + 0.5170 + 0.5107 \\
&= 1.7219 \text{ bits}
\end{aligned} \tag{8-42}$$

The entropy is definitely less than 2 bits. It is slightly over 1.5 bits, which would be the entropy if the four probabilities were rounded to the nearest 0.25.

**Solution to Problem 1, part l.**

The average cost per meal is given by the equation:

$$\begin{aligned}
\$1B + \$2C + \$3F + \$8T &= \$1 \times 0.0643 + \$2 \times 0.1746 + \$3 \times 0.2875 + \$8 \times 0.4736 \\
&= \$0.0643 + \$0.3492 + \$0.8625 + \$3.7888 \\
&= \$5.07
\end{aligned} \tag{8-43}$$

**Solution to Problem 1, part m.**

The entropy is far enough from 2 bits that a more efficient encoding is worth considering. A Huffman code which considers single orders at a time (say 0 for  $T$ , 10 for  $F$ , 110 for  $C$ , and 011 for  $B$ ) would have an average length of 1.7653 bits per symbol. This is a saving of about 12%. If sequences of orders are encoded together, there could be a savings of no more than about 14%.

On the negative side, the system would cost more to build and maintain, and because of its greater complexity it would probably fail more often and take longer to repair. If customer preferences changed over the years, the entropy might rise and the communication channel would not be able to handle the new load, or the coding might have to change to accommodate the new probabilities. Another point is that a block code takes advantage of the lower entropy by bunching together many meal orders (perhaps 5 in this case) and would not send a single bit to the kitchen until 5 customers had arrived and placed their orders. The restaurants already have a low rating in the [Zagat Surveys](#) because so many meals arrive cold, and any latency in order fulfillment would only make matters worse.